CHAPTER 2

THE MATHEMATICS OF OPTIMIZATION

The problems in this chapter are primarily mathematical. They are intended to give students some practice with taking derivatives and using the Lagrangian techniques, but the problems in themselves offer few economic insights. Consequently, no commentary is provided. All of the problems are relatively simple and instructors might choose from among them on the basis of how they wish to approach the teaching of the optimization methods in class.

Solutions

2.1 $U(x,y) = 4x^2 + 3y^2$

a. $\frac{\partial U}{\partial x} = 8x$, $\frac{\partial U}{\partial y} = 6y$

b. 8, 12

c. $dU = \frac{\partial U}{\partial x}\, dx + \frac{\partial U}{\partial y}\, dy = 8x\ dx + 6y\ dy$

d. $\frac{dy}{dx}$ for $dU = 0$ $\quad 8x\ dx + 6y\ dy = 0$

$$\frac{dy}{dx} = \frac{-8x}{6y} = \frac{-4x}{3y}$$

e. $x = 1$, $\quad y = 2 \quad U = 4 \cdot 1 + 3 \cdot 4 = 16$

f. $\frac{dy}{dx} = \frac{-4(1)}{3(2)} = -2/3$

g. U = 16 contour line is an ellipse centered at the origin. With equation

$4x^2 + 3y^2 = 16$, slope of the line at (x, y) is $\frac{dy}{dx} = -\frac{4x}{3y}$.

2.2 a. Profits are given by $\pi = R - C = -2q^2 + 40q - 100$

$\frac{d\pi}{dq} = -4q + 40 \quad q^* = 10$

$\pi^* = -2(10)^2 + 40(10) - 100 = 100$

b. $\frac{d^2\pi}{dq^2} = -4$ so profits are maximized

c. $MR = \frac{dR}{dq} = 70 - 2q$

$$MC = \frac{dC}{dq} = 2q + 30$$

so $q^* = 10$ obeys $MR = MC = 50$.

2.3 Substitution: $y = 1 - x$ so $f = xy = x - x^2$

$$\frac{\partial f}{\partial x} = 1 - 2x = 0$$

$x = 0.5, y = 0.5, f = 0.25$

Note: $f'' = -2 < 0$. This is a local and global maximum.

Lagrangian Method: $? = xy + \lambda\ 1 - x - y)$

$$\frac{\partial £}{\partial x} = y - \lambda = 0$$

$$\frac{\partial £}{\partial y} = x - \lambda = 0$$

so, $x = y$.

using the constraint gives $x = y = 0.5, \quad xy = 0.25$

2.4 Setting up the Lagrangian: $? = x + y + \lambda\ 0.25 - xy)$.

$$\frac{\partial £}{\partial x} = 1 - \lambda y$$

$$\frac{\partial £}{\partial y} = 1 - \lambda x$$

So, $x = y$. Using the constraint gives $xy = x^2 = 0.25, \quad x = y = 0.5$.

2.5 a. $f(t) = -0.5gt^2 + 40t$

$$\frac{df}{dt} = -g\,t + 40 = 0, \quad t^* = \frac{40}{g}\ .$$

b. Substituting for t^*, $f(t^*) = -0.5g(40/g)^2 + 40(40/g) = 800/g$.

$$\frac{\partial f(t^*)}{\partial g} = -800/g^2\ .$$

c. $\frac{\partial f}{\partial g} = -\frac{1}{2}(t^*)^2$ depends on g because t^* depends on g.

so $\frac{\partial f}{\partial g} = -0.5(t^*)^2 = -0.5(\frac{40}{g})^2 = \frac{-800}{g^2}$.

d. $800/32 = 25, \quad 800/32.1 = 24.92$, a reduction of .08. Notice that $-800/g^2 = 800/32^2 \approx -0.8$ so a 0.1 increase in g could be predicted to reduce height by 0.08 from the envelope theorem.

2.6 a. This is the volume of a rectangular solid made from a piece of metal which is x by $3x$ with the defined corner squares removed.

b. $\frac{\partial V}{\partial t} = 3x^2 - 16xt + 12t^2 = 0$. Applying the quadratic formula to this expression yields

$$t = \frac{16x \pm \sqrt{256x^2 - 144x^2}}{24} = \frac{16x \pm 10.6x}{24} = 0.225x, \quad 1.11x.$$

To determine true maximum must look at second derivative -- $\frac{\partial^2 V}{\partial t^2} = -16x + 24t$ which is negative only for the first solution.

c. If $t = 0.225x, \quad V \approx 0.67x^3 - .04x^3 + .05x^3 \approx 0.68x^3$ so V increases without limit.

d. This would require a solution using the Lagrangian method. The optimal solution requires solving three non-linear simultaneous equations—a task not undertaken here. But it seems clear that the solution would involve a different relationship between t and x than in parts a-c.

2.7 a. Set up Lagrangian $? = x_1 + \quad \ln x_2 + \lambda(k - x_1 - x_2)$ yields the first order conditions:

$$\frac{\partial \pounds}{\partial x_1} = 1 - \lambda = 0$$

$$\frac{\partial ?}{\partial x_2} = \frac{\quad}{x_2} - \lambda = 0$$

$$\frac{\partial \pounds}{\partial \lambda} = k - x_1 - x_2 = 0$$

Hence, $\lambda = 1 = 5/x_2$ or $x_2 = 5$. With $k = 10$, optimal solution is $x_1 = x_2 = 5$.

b. With $k = 4$, solving the first order conditions yields $x_2 = 5, x_1 = -1$.

c. Optimal solution is $x_1 = 0, x_2 = 4, y = 5\ln 4$. Any positive value for x_1 reduces y.

d. If $k = 20$, optimal solution is $x_1 = 15, x_2 = 5$. Because x_2 provides a diminishing marginal increment to y whereas x_1 does not, all optimal solutions require that, once x_2 reaches 5, any extra amounts be devoted entirely to x_1.

2.8 The proof is most easily accomplished through the use of the matrix algebra of quadratic forms. See, for example, Mas Colell et al., pp. 937–939. Intuitively, because concave functions lie below any tangent plane, their level curves must also be convex. But the converse is not true. Quasi-concave functions may exhibit “increasing returns to scale”; even though their level curves are convex, they may rise above the tangent plane when all variables are increased together.

2.9 a. $f_1 = \alpha\, x_1^{\alpha-1} x_2^{\beta} > 0.$

$f_2 = \beta\, x_1^{\alpha} x_2^{\beta-1} > 0.$

$f_{11} = \alpha(\alpha - 1)\ x_1^{\alpha-2}\ x_1^{\beta} < 0.$

$f_{22} = \beta(\beta - 1)\ x_1^{\alpha}\ x_2^{\beta-2} < 0.$

$f_{12} = f_{21} = \alpha\,\beta\, x_1^{\alpha-1} x_2^{\beta-1} > 0.$

Clearly, all the terms in Equation 2.114 are negative.

b. If $y = c = x_1^{\alpha}\ x_2^{\beta}$

$x_2 = c^{1/\beta}\ x_1^{-\alpha/\beta}$ since α, β > 0, x_2 is a convex function of x_1.

c. Using equation 2.98,

$f_{11} f_{22} - f_{12}^2 = \alpha(\alpha - 1)(\beta)(\beta - 1)\ x_1^{2\alpha-2}\ x_2^{2\beta-2} - \alpha^2 \beta^2 x_1^{2\alpha-2} x_2^{2\beta-2}$

$= \alpha\,\beta(1 - \beta - \alpha)\, x_1^{2\alpha-2} x_2^{2\beta-2}$ which is negative for α + β > 1.

2.10 a. Since $y' > 0, y'' < 0$, the function is concave.

b. Because $f_{11}, f_{22} < 0$, and $f_{12} = f_{21} = 0$, Equation 2.98 is satisfied and the function is concave.

c. y is quasi-concave as is y^{γ}. But y^{γ} is not concave for $\gamma > 1$. All of these results can be shown by applying the various definitions to the partial derivatives of y.

CHAPTER 3

PREFERENCES AND UTILITY

These problems provide some practice in examining utility functions by looking at indifference curve maps. The primary focus is on illustrating the notion of a diminishing MRS in various contexts. The concepts of the budget constraint and utility maximization are not used until the next chapter.

Comments on Problems

3.1 This problem requires students to graph indifference curves for a variety of functions, some of which do not exhibit a diminishing *MRS*.

3.2 Introduces the formal definition of quasi-concavity (from Chapter 2) to be applied to the functions in Problem 3.1.

3.3 This problem shows that diminishing marginal utility is not required to obtain a diminishing *MRS*. All of the functions are monotonic transformations of one another, so this problem illustrates that diminishing *MRS* is preserved by monotonic transformations, but diminishing marginal utility is not.

3.4 This problem focuses on whether some simple utility functions exhibit convex indifference curves.

3.5 This problem is an exploration of the fixed-proportions utility function. The problem also shows how such problems can be treated as a composite commodity.

3.6 In this problem students are asked to provide a formal, utility-based explanation for a variety of advertising slogans. The purpose is to get students to think mathematically about everyday expressions.

3.7 This problem shows how initial endowments can be incorporated into utility theory.

3.8 This problem offers a further exploration of the Cobb-Douglas function. Part c provides an introduction to the linear expenditure system. This application is treated in more detail in the Extensions to Chapter 4.

3.9 This problem shows that independent marginal utilities illustrate one situation in which diminishing marginal utility ensures a diminishing *MRS*.

3.10 This problem explores various features of the *CES* function with weighting on the two goods.

Solutions

3.1 Here we calculate the MRS for each of these functions:

a. $MRS = f_x / f_y = 3/1$ — *MRS* is constant.

b. $MRS = f_x \Big/ f_y = \dfrac{0.5(y/x)^{0.5}}{0.5(y/x)^{-0.5}} = y/x$ — *MRS* is diminishing.

c. $MRS = f_x / f_y = 0.5x^{-0.5}/1$ — *MRS* is diminishing

d. $MRS = f_x / f_y = 0.5(x^2 - y^2)^{-0.5} \cdot 2x / 0.5(x^2 - y^2)^{-0.5} \cdot 2y = x/y$ — *MRS* is increasing.

e. $MRS = f_x \Big/ f_y = \dfrac{(x+y)y - xy}{(x+y)^2} \Big/ \dfrac{(x+y)x - xy}{(x+y)^2} = y^2/x^2$ — *MRS* is diminishing.

3.2 Because all of the first order partials are positive, we must only check the second order partials.

a. $f_{11} = f_{22} = f_2 = 0$ Not strictly quasiconcave.

b. $f_{11}, f_{22} < 0,\ f_{12} > 0$ Strictly quasiconcave

c. $f_{11} < 0,\ f_{22} = 0,\ f_{12} = 0$ Strictly quasiconcave

d. Even if we only consider cases where $x \geq y$, both of the own second order partials are ambiguous and therefore the function is not necessarily strictly quasiconcave.

e. $f_{11}, f_{22} < 0\ f_{12} > 0$ Strictly quasiconcave.

3.3 a. $U_x = y, U_{xx} = 0, U_y = x, U_{yy} = 0, MRS = y/x$.

b. $U_x = 2xy^2, U_{xx} = 2y^2, U_y = 2x^2y, U_{yy} = 2x^2, MRS = y/x$.

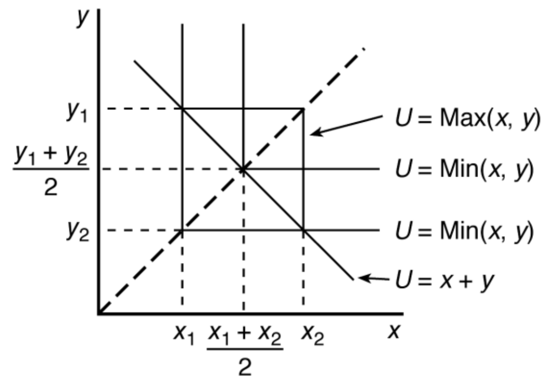
c. $U_x = 1/x, U_{xx} = -1/x^2, U_y = 1/y, U_{yy} = -1/y^2, MRS = y/x$

This shows that monotonic transformations may affect diminishing marginal utility, but not the *MRS*.

3.4 a. The case where the same good is limiting is uninteresting because $U(x_1, y_1) = x_1 = k = U(x_2, y_2) = x_2 = U[(x_1 + x_2)/2, (y_1 + y_2)/2] = (x_1 + x_2)/2$. If the limiting goods differ, then $y_1 > x_1 = k = y_2 < x_2$. Hence, $(x_1 + x_2)/2 > k$ and $(y_1 + y_2)/2 > k$ so the indifference curve is convex.

b. Again, the case where the same good is maximum is uninteresting. If the goods differ, $y_1 < x_1 = k = y_2 > x_2$. $(x_1 + x_2)/2 < k$, $(y_1 + y_2)/2 < k$ so the indifference curve is concave, not convex.

c. Here $(x_1 + y_1) = k = (x_2 + y_2) = [(x_1 + x_2)/2, (y_1 + y_2)/2]$ so indifference curve is neither convex or concave – it is linear.



3.5 a. $U(h,b,m,r) = Min(h, 2b, m, 0.5r)$.

b. A fully condimented hot dog.

c. $1.60

d. $2.10 – an increase of 31 percent.

e. Price would increase only to $1.725 – an increase of 7.8 percent.

f. Raise prices so that a fully condimented hot dog rises in price to $2.60. This would be equivalent to a lump-sum reduction in purchasing power.

3.6 a. $U(p,b) = p + b$

b. $\frac{\partial^2 U}{\partial x \partial coke} > 0.$

c. $U(p,x) > U(1,x)$ for $p > 1$ and all x.

d. $U(k,x) > U(d,x)$ for $k = d$.

e. See the extensions to Chapter 3.

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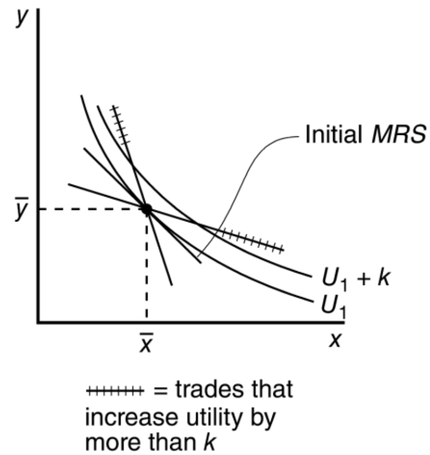

b. Any trading opportunities that differ from the *MRS* at $\bar{x}, \bar{y}$ will provide the opportunity to raise utility (see figure).

c. A preference for the initial endowment will require that trading opportunities raise utility substantially. This will be more likely if the trading opportunities and significantly different from the initial *MRS* (see figure).

3.8 a. $MRS = \dfrac{\partial U / \partial x}{\partial U / \partial y} = \dfrac{\alpha x^{\alpha-1} y^{\beta}}{\beta x^{\alpha} \; y^{\beta-1}} = \dfrac{\alpha}{\beta} \; (y/x)$

This result does not depend on the sum α + β which, contrary to production theory, has no significance in consumer theory because utility is unique only up to a monotonic transformation.

b. Mathematics follows directly from part a. If α > β the individual values *x* relatively more highly; hence, $dy/dx > 1$ for *x* = *y*.

c. The function is homothetic in $(x - x_0)$ and $(y - y_0)$, but not in *x* and *y*.

3.9 From problem 3.2, $f_{12} = 0$ implies diminishing MRS providing $f_{11}, f_{22} < 0$.
Conversely, the Cobb-Douglas has $f_{12} > 0, f_{11}, f_{22} < 0$, but also has a diminishing *MRS* (see problem 3.8a).

3.10 a. $MRS = \dfrac{\partial U / \partial x}{\partial U / \partial y} = \dfrac{\alpha x^{\delta-1}}{\beta y^{\delta-1}} = \dfrac{\alpha}{\beta} \; (y/x)^{1-\delta}$ so this function is homothetic.

b. If δ = 1, *MRS* = α/β, a constant.
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c. For δ < 1 1 – δ > 0, so *MRS* diminishes.

d. Follows from part a, if *x* = *y* *MRS* = α/β.

e. With $\delta = .5, MRS(.9) = \frac{\alpha}{\beta}(.9)^{0.5} = .949\frac{\alpha}{\beta}$

$MRS\ (1.1) = \frac{\alpha}{\beta}\ (1.1)^{0.5} = 1.05\ \frac{\alpha}{\beta}$

With $\delta = -1, MRS(.9) = \frac{\alpha}{\beta}(.9)^2 = .81\ \frac{\alpha}{\beta}$

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Hence, the *MRS* changes more dramatically when δ = –1 than when δ = .5; the lower δ is, the more sharply curved are the indifference curves. When $\delta = -\infty$, the indifference curves are L-shaped implying fixed proportions.

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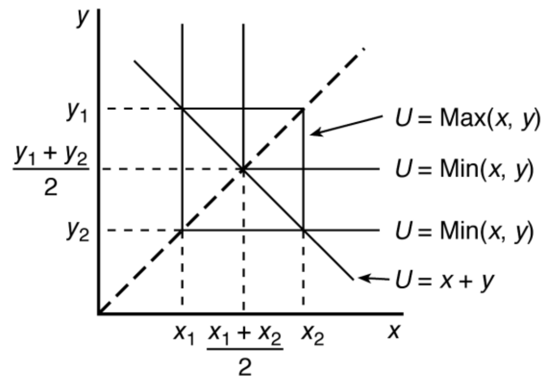

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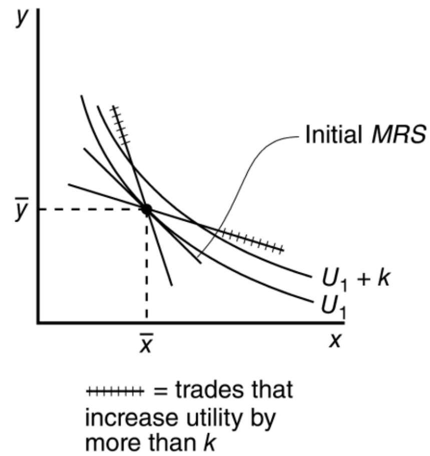

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Hence, the *MRS* changes more dramatically when $\delta = -1$ than when $\delta = .5$; the lower δ is, the more sharply curved are the indifference curves. When $\delta = -\infty$, the indifference curves are L-shaped implying fixed proportions.

CHAPTER 4

UTILITY MAXIMIZATION AND CHOICE

The problems in this chapter focus mainly on the utility maximization assumption. Relatively simple computational problems (mainly based on Cobb–Douglas and CES utility functions) are included. Comparative statics exercises are included in a few problems, but for the most part, introduction of this material is delayed until Chapters 5 and 6.

Comments on Problems

4.1 This is a simple Cobb–Douglas example. Part (b) asks students to compute income compensation for a price rise and may prove difficult for them. As a hint they might be told to find the correct bundle on the original indifference curve first, then compute its cost.

4.2 This uses the Cobb-Douglas utility function to solve for quantity demanded at two different prices. Instructors may wish to introduce the expenditure shares interpretation of the function's exponents (these are covered extensively in the Extensions to Chapter 4 and in a variety of numerical examples in Chapter 5).

4.3 This starts as an unconstrained maximization problem—there is no income constraint in part (a) on the assumption that this constraint is not limiting. In part (b) there is a total quantity constraint. Students should be asked to interpret what Lagrangian Multiplier means in this case.

4.4 This problem shows that with concave indifference curves first order conditions do not ensure a local maximum.

4.5 This is an example of a “fixed proportion” utility function. The problem might be used to illustrate the notion of perfect complements and the absence of relative price effects for them. Students may need some help with the min ( ) functional notation by using illustrative numerical values for v and g and showing what it means to have “excess” v or g.

4.6 This problem introduces a third good for which optimal consumption is zero until income reaches a certain level.

4.7 This problem provides more practice with the Cobb-Douglas function by asking students to compute the indirect utility function and expenditure function in this case. The manipulations here are often quite difficult for students, primarily because they do not keep an eye on what the final goal is.

4.8 This problem repeats the lessons of the lump sum principle for the case of a subsidy. Numerical examples are based on the Cobb-Douglas expenditure function.

4.9 This problem looks in detail at the first order conditions for a utility maximum with the *CES* function. Part c of the problem focuses on how relative expenditure shares are determined with the *CES* function.

4.10 This problem shows utility maximization in the linear expenditure system (see also the Extensions to Chapter 4).

Solutions

4.1 a. Set up Lagrangian

$$? = \sqrt{ts} + \lambda\ 1.00 - .10t - .25s) .$$

$$\frac{\partial £}{\partial t} = (s/t)^{0.5} - .10\lambda$$

$$\frac{\partial £}{\partial s} = (t/s)^{0.5} - .25\lambda$$

$$\frac{\partial £}{\partial \lambda} = 1.00 - .10t - .25s = 0$$

Ratio of first two equations implies

$$\frac{t}{s} = 2.5 \quad t = 2.5s$$

Hence

1.00 = .10t + .25s = .50s.

s = 2 t = 5

Utility = $\sqrt{10}$

b. New utility $\sqrt{10}$ or ts = 10

and $\frac{t}{s} = \frac{.25}{.40} = \frac{5}{8}$

$$t = \frac{5s}{8}$$

Substituting into indifference curve:

$$\frac{5s^2}{8} = 10$$

s^2 = 16 s = 4 t = 2.5

Cost of this bundle is 2.00, so Paul needs another dollar.

4.2 Use a simpler notation for this solution: $U(f,c) = f^{2/3}c^{1/3} \quad I = 300$

a. $? = f^{2/3}c^{1/3} + \lambda \;\; 300 - 20f - 4c)$

$$\frac{\partial £}{\partial f} = 2/3(c/f)^{1/3} - 20\lambda$$

$$\frac{\partial £}{\partial c} = 1/3(f/c)^{2/3} - 4\lambda$$

Hence,

$$5 = 2 \;\; \frac{c}{f}, 2c = 5f$$

Substitution into budget constraint yields f = 10, c = 25.

b. With the new constraint: f = 20, c = 25

Note: This person always spends 2/3 of income on f and 1/3 on c. Consumption of California wine does not change when price of French wine changes.

c. In part a, $U(f,c) = f^{2/3}c^{1/3} = 10^{2/3}25^{1/3} = 13.5$. In part b, $U(f,c) = 20^{2/3}25^{1/3} = 21.5$. To achieve the part b utility with part a prices, this person will need more income. Indirect utility is $21.5 = (2/3)^{2/3}(1/3)^{1/3} I p_f^{-2/3} p_c^{-1/3} = (2/3)^{2/3} I 20^{-2/3} 4^{-1/3}$. Solving this equation for the required income gives I = 482. With such an income, this person would purchase f = 16.1, c = 40.1, U = 21.5.

4.3 $U(c,b) = 20c - c^2 + 18b - 3b^2$

a. $\frac{\partial U}{\partial c} = 20 - 2c = 0, \quad c = 10 \;|$

$$\frac{\partial U}{\partial b} = 18 - 6b = 0, \quad b = 3$$

So, U = 127.

b. Constraint: $b + c = 5$

$$? = \;\; 0c - c^2 + 18b - 3b^2 + \lambda(5 - c - b)$$

$$\frac{\partial £}{\partial c} = 20 - 2c - \lambda = 0$$

$$\frac{\partial £}{\partial b} = 18 - 6b - \lambda = 0$$

$$\frac{\partial £}{\partial \lambda} = 5 - c - b = 0$$

$c = 3b + 1$ so $b + 3b + 1 = 5$, $b = 1$, $c = 4$, $U = 79$

4.4 $U(x, y) = (x^2 + y^2)^{0.5}$

Maximizing U^2 in will also maximize U.

a. $? = x^2 + y^2 + \lambda\ 50 - 3x - 4y)$

$$\frac{\partial £}{\partial x} = 2x - 3\lambda = 0 \qquad \lambda = 2x/3$$

$$\frac{\partial £}{\partial y} = 2y - 4\lambda = 0 \qquad \lambda = y/2$$

$$\frac{\partial £}{\partial \lambda} = 50 - 3x - 4y = 0$$

First two equations give $y = 4x/3$. Substituting in budget constraint gives x = 6, y = 8 , U = 10.

b. This is not a local maximum because the indifference curves do not have a diminishing MRS (they are in fact concentric circles). Hence, we have necessary but not sufficient conditions for a maximum. In fact the calculated allocation is a minimum utility. If Mr. Ball spends all income on x, say, U = 50/3.

4.5 $U(m) = U(g, v) = Min[g/2, v]$

a. No matter what the relative price are (i.e., the slope of the budget constraint) the maximum utility intersection will always be at the vertex of an indifference curve where $g = 2v$.

b. Substituting $g = 2v$ into the budget constraint yields:

$$2p_g v + p_v v = I \text{ or } v = \frac{I}{2p_g + p_v} \ .$$

Similarly, $g = \dfrac{2I}{2p_g + p_v}$

It is easy to show that these two demand functions are homogeneous of degree zero in P_G , P_V , and I.

c. $U = g/2 = v$ so,

Indirect Utility is $V(p_g, p_v, I) = \dfrac{I}{2p_g + p_v}$

d. The expenditure function is found by interchanging I (= E) and V, $E(p_g, p_v, V) = (2p_g + p_v)V$.

4.6 a. If $x = 4$ $y = 1$ $U(z = 0) = 2$.

If $z = 1$ $U = 0$ since $x = y = 0$.

If $z = 0.1$ (say) $x = .9/.25 = 3.6$, $y = .9$.

$U = (3.6)^{.5}(.9)^{.5}(1.1)^{.5} = 1.89$ – which is less than $U(z = 0)$

b. At $x = 4$ $y = 1$ $z = 0$

$MU_x / p_x = MU_y / p_y = 1$

$MU_z / p_z = 1/2$

So, even at $z = 0$, the marginal utility from z is "not worth" the good's price. Notice here that the "1" in the utility function causes this individual to incur some diminishing marginal utility for z before any is bought. Good z illustrates the principle of "complementary slackness discussed in Chapter 2.

c. If $I = 10$, optimal choices are $x = 16$, $y = 4$, $z = 1$. A higher income makes it possible to consume z as part of a utility maximum. To find the minimal income at which any (fractional) z would be bought, use the fact that with the Cobb-Douglas this person will spend equal amounts on x, y, and (1+z). That is:

$$p_x x = p_y y = p_z(1+z)$$

Substituting this into the budget constraint yields:

$$2p_z(1+z) + p_z z = I \quad or \quad 3p_z z = I - 2p_z$$

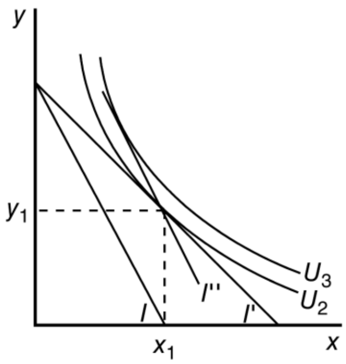
Hence, for $z > 0$ it must be the case that $I > 2p_z$ or $I > 4$.

4.7 $U(x,y) = x^{\alpha} y^{1-\alpha}$

a. The demand functions in this case are $x = \alpha I/p_x$, $y = (1-\alpha)I/p_y$. Substituting these into the utility function gives $V(p_x, p_y, I) = [\alpha I/p_x]^{\alpha}[(1-\alpha)I/p_y] = BIp_x^{-\alpha} p_y^{-(1-\alpha)}$ where $B = \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}$.

b. Interchanging I and V yields $E(p_x, p_y, V) = B^{-1} p_x^{\alpha} p_y^{(1-\alpha)} V$.

c. The elasticity of expenditures with respect to p_x is given by the exponent α. That is, the more important x is in the utility function the greater the proportion that expenditures must be increased to compensate for a proportional rise in the price of x.



4.8 a.

b. $E(p_x, p_y, U) = 2p_x^{0.5} p_y^{0.5} U$. With $p_x = 1, p_y = 4, U = 2, E = 8$. To raise utility to 3 would require E = 12 – that is, an income subsidy of 4.

c. Now we require $E = 8 = 2p_x^{0.5} 4^{0.5} 3$ or $p_x^{0.5} = 8/12 = 2/3$. So $p_x = 4/9$ -- that is, each unit must be subsidized by 5/9. at the subsidized price this person chooses to buy x = 9. So total subsidy is 5 – one dollar greater than in part c.

d. $E(p_x, p_y, U) = 1.84 p_x^{0.3} p_y^{0.7} U$. With $p_x = 1, p_y = 4, U = 2, E = 9.71$. Raising U to 3 would require extra expenditures of 4.86. Subsidizing good x alone would require a price of $p_x = 0.26$. That is, a subsidy of 0.74 per unit. With this low price, this person would choose x = 11.2, so total subsidy would be 8.29.

4.9 a. $MRS = \dfrac{\partial U/\partial x}{\partial U/\partial y} = (x/y)^{\delta - 1} = p_x/p_y$ for utility maximization.

Hence, $x/y = (p_x/p_y)^{1/(\delta-1)} = (p_x/p_y)^{-\sigma}$ where $\sigma = 1/(1-\delta)$.

b. If δ = 0, $x/y = p_y/p_x$ so $p_x x = p_y y$.

c. Part a shows $p_x x/p_y y = (p_x/p_y)^{1-\sigma}$

Hence, for $\sigma < 1$ the relative share of income devoted to good x is positively correlated with its relative price. This is a sign of low substitutability. For $\sigma > 1$ the relative share of income devoted to good x is negatively correlated with its relative price – a sign of high substitutability.

d. The algebra here is very messy. For a solution see the Sydsaeter, Strom, and Berck reference at the end of Chapter 5.

4.10 a. For $x < x_0$ utility is negative so will spend $p_x x_0$ first. With $I - p_x x_0$ extra income, this is a standard Cobb-Douglas problem:

$$p_x(x - x_0) = \alpha(I - p_x x_0), \quad p_y y = \beta(I - p_x x_0)$$

b. Calculating budget shares from part a yields

$$\frac{p_x x}{I} = \alpha + \frac{(1-\alpha) p_x x_0}{I}, \quad \frac{p_y y}{I} = \beta - \frac{\beta p_x x_0}{I}$$

$$\lim(I \to \infty) \frac{p_x x}{I} = \alpha, \lim(I \to \infty) \frac{p_y y}{I} = \beta.$$

CHAPTER 5

INCOME AND SUBSTITUTION EFFECTS

Problems in this chapter focus on comparative statics analyses of income and own-price changes. Many of the problems are fairly easy so that students can approach the ideas involved in shifting budget constraints in simplified settings. Theoretical material is confined mainly to the Extensions where Shephard's Lemma and Roy's Identity are illustrated for the Cobb-Douglas case.

Comments on Problems

5.1 An example of perfect substitutes.

5.2 A fixed-proportions example. Illustrates how the goods used in fixed proportions (peanut butter and jelly) can be treated as a single good in the comparative statics of utility maximization.

5.3 An exploration of the notion of homothetic functions. This problem shows that Giffen's Paradox cannot occur with homothetic functions.

5.4 This problem asks students to pursue the analysis of Example 5.1 to obtain compensated demand functions. The analysis essentially duplicates Examples 5.3 and 5.4.

5.5 Another utility maximization example. In this case, utility is not separable and cross-price effects are important.

5.6 This is a problem focusing on "share elasticities". It shows that more customary elasticities can often be calculated from share elasticities—this is important in empirical work where share elasticities are often used.

5.7 This is a problem with no substitution effects. It shows how price elasticities are determined only by income effects which in turn depend on income shares.

5.8 This problem illustrates a few simple cases where elasticities are directly related to parameters of the utility function.

5.9 This problem shows how the aggregation relationships described in Chapter 5 for the case of two goods can be generalized to many goods.

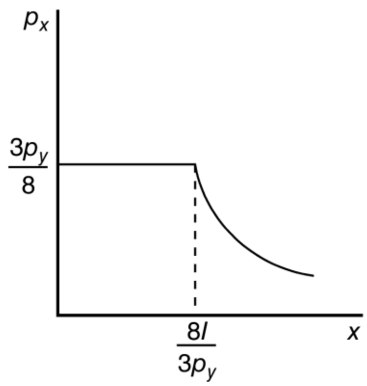
5.10 A revealed preference example of inconsistent preferences.

Solutions

5.1 a. Utility = Quantity of water = $.75x + 2y$.

b. If $p_x < \frac{3}{8}\ p_y \quad x = I/p_x, y = 0$.

If $p_x > \frac{3}{8} p_y \quad x = 0,\ y = \frac{I}{p_y}$.

c. 

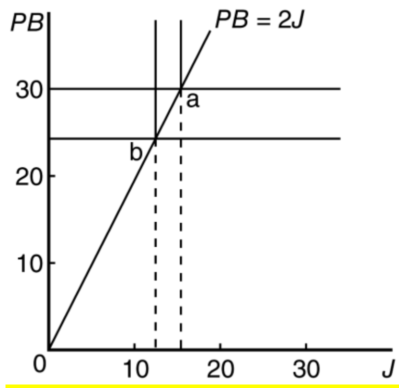
d. Increases in I shifts demand for x outward. Reductions in p_y do not affect demand for x until $p_y < \frac{8p_x}{3}$. Then demand for x falls to zero.

e. The income-compensated demand curve for good x is the single x, p_x point that characterizes current consumption. Any change in p_x would change utility from this point (assuming $x > 0$).

5.2 a. Utility maximization requires $pb = 2j$ and the budget constraint is $.05pb + .1j = 3$. Substitution gives $pb = 30$, $j = 15$

b. If $p_j = \$.15$ substitution now yields $j = 12$, $pb = 24$.

c. To continue buying $j = 15$, $pb = 30$, David would need to buy 3 more ounces of jelly and 6 more ounces of peanut butter. This would require an increase in income of: $3(.15) + 6(.05) = .75$.

d. 

e. Since David N. uses only $pb + j$ to make sandwiches (in fixed proportions), and because bread is free, it is just as though he buys sandwiches where $p_{\text{sandwich}} = 2p_{pb} + p_j$.

In part a, $p_s = .20$, $q_s = 15$;

In part b, $p_s = .25$, $q_s = 12$;

In general, $q_s = \dfrac{3}{p_s}$ so the demand curve for sandwiches is a hyperbola.

f. There is no substitution effect due to the fixed proportion. A change in price results in only an income effect.

5.3 a. As income increases, the ratio p_x/p_y stays constant, and the utility-maximization conditions therefore require that *MRS* stay constant. Thus, if *MRS* depends on the ratio y/x, this ratio must stay constant as income increases. Therefore, since income is spent only on these two goods, both *x* and *y* are proportional to income.

b. Because of part (a), $\dfrac{\partial x}{\partial I} > 0$ so Giffen's paradox cannot arise.

5.4 a. Since $x = 0.3I/p_x, \quad y = 0.7I/p_y$,

$$U = .3^{.3}.7^{.7}\,Ip_x^{-.3}p_y^{-.7} = BIp_x^{-.3}p_y^{-.7}$$

The expenditure function is then $E = B^{-1}Up_x^{.3}p_y^{.7}$.

b. The compensated demand function is $x^c = \partial E/\partial p_x = .3B^{-1}p_x^{-.7}p_y^{.7}$.

c. It is easiest to show Slutsky Equation in elasticities by just reading exponents from the various demand functions: $e_{x,p_x} = -1, e_{x,I} = 1, e_{x^c,p_x} = -.7, s_x = 0.3$

Hence $e_{x,p_x} = e_{x^c,p_x} - s_x e_{x,I}$ or $-1 = -0.7 - 0.3 \cdot 1$

5.6 a. $e_{s_x,I} = \dfrac{\partial\, p_x x/I}{\partial I}\cdot\dfrac{I}{p_x x/I} = \dfrac{Ip_x\,\partial x/\partial I - p_x x}{I^2}\cdot\dfrac{I^2}{p_x x} = e_{x,I} - 1$.

If, for example $e_{x,I} = 1.5,\quad e_{s_x,I} = 0.5$.

b. $e_{s_x,p_x} = \dfrac{\partial(p_x x/I)}{\partial p_x}\cdot\dfrac{p_x}{p_x x/I} = \dfrac{p_x\,\partial x/\partial p_x + x}{I}\cdot\dfrac{I}{x} = e_{x,p_x} + 1$

If, for example, $e_{x,p_x} = -0.75,\quad e_{s_x,p_x} = +0.25$.

c. Because *I* may be cancelled out of the derivation in part b, it is also the case that $e_{p_x x,p_x} = e_{x,p_x} + 1$.

d. $e_{s_x,p_y} = \dfrac{\partial(p_x x/I)}{\partial p_y}\cdot\dfrac{p_y}{p_x x/I} = \dfrac{p_x\,\partial x/\partial p_y}{I}\cdot\dfrac{p_y I}{p_x x} = \dfrac{\partial x}{\partial p_y}\cdot\dfrac{p_y}{x} = e_{x,p_y}$.

e. Use part b: $e_{s_x,p_x} = \dfrac{kp_y^k p_x^{-k-1}}{(1+p_y^k p_x^{-k})^2}\cdot p_x(1+p_y^k p_x^{-k}) = \dfrac{kp_y^k p_x^{-k}}{1+p_y^k p_x^{-k}}$.

To simplify algebra, let $d = p_y^k p_x^{-k}$

Hence $e_{x,p_x} = e_{s_x,p_x} - 1 = \dfrac{kd}{1+d} - 1 = \dfrac{kd - 1 - d}{1+d}$. Now use the Slutsky equation, remembering that $e_{x,I} = 1$.

$$e_{x^c,p_x} = e_{x,p_x} + s_x = \frac{kd - d - 1}{1+d} + \frac{1}{1+d} = \frac{d(k-1)}{1+d} = (1-s_x)(-\sigma)\ .$$

5.7 a. Because of the fixed proportions between *h* and *c*, know that the demand for ham is $h = I/(p_h + p_c)$. Hence

$$e_{h,p_h} = \frac{\partial h}{\partial p_h}\cdot\frac{p_h}{h} = \frac{-I}{(p_h+p_c)^2}\cdot\frac{p_h(p_h+p_c)}{I} = \frac{-p_h}{(p_h+p_c)}\ .$$

Similar algebra shows that $e_{h,p_c} = \dfrac{-p_c}{(p_h+p_c)}$. So, if $p_h = p_c$, $e_{h,p_h} = e_{h,p_c} = -0.5$.

b. With fixed proportions there are no substitution effects. Here the compensated price elasticities are zero, so the Slutsky equation shows that $e_{x,p_x} = 0 - s_x = -0.5$.

c. With $p_h = 2p_c$ part a shows that $e_{h,p_h} = \dfrac{-2}{3}$, $e_{h,p_c} = \dfrac{-1}{3}$.

d. If this person consumes only ham and cheese sandwiches, the price elasticity of demand for those must be -1. Price elasticity for the components reflects the proportional effect of a change in the price of the component on the price the whole sandwich. In part a, for example, a ten percent increase in the price of ham will increase the price of a sandwich by 5 percent and that will cause quantity demanded to fall by 5 percent.

5.8 a. $e_{x,p_x} = -(1-s_x)\sigma - s_x \quad e_{y,p_y} = -s_x\sigma - s_y$ Hence $e_{x,p_x} + e_{y,p_y} = -\sigma - 1$.

The sum equals -2 (trivially) in the Cobb-Douglas case.

b. Result follows directly from part a. Intuitively, price elasticities are large when σ is large and small when σ is small.

c. A generalization from the multivariable *CES* function is possible, but the constraints placed on behavior by this function are probably not tenable.

5.9 a. Because the demand for any good is homogeneous of degree zero, Euler's theorem states $\sum_{j=1}^{n} p_j \frac{\partial x_i}{\partial p_j} + I\frac{\partial x_i}{\partial I} = 0$.

Multiplication by $1/x_i$ yields the desired result.

b. Part b and c are based on the budget constraint $\sum_i p_i x_i = I$.

Differentiation with respect to *I* yields: $\sum_i p_i \, \partial x_i / \partial I = 1$.

Multiplication of each term by $x_i I / x_i I$ yields $\sum_i s_i e_{i,I} = 1$.

c. Differentiation of the budget constraint with respect to p_j :

$\sum_i p_i \, \partial x_i / \partial p_j + x_j = 0$. Multiplication by $\frac{p_j}{I} \cdot \frac{x_i}{x_i}$ yields

$\sum_i s_i e_{i,j} = -s_j$.

5.10 Year 2's bundle is revealed preferred to Year 1's since both cost the same in Year 2's prices. Year 2's bundle is also revealed preferred to Year 3's for the same reason. But in Year 3, Year 2's bundle costs less than Year 3's but is not chosen. Hence, these violate the axiom.

CHAPTER 6

DEMAND RELATIONSHIPS AMONG GOODS

Two types of demand relationships are stressed in the problems to Chapter 6: cross-price effects and composite commodity results. The general goal of these problems is to illustrate how the demand for one particular good is affected by economic changes that directly affect some other portion of the budget constraint. Several examples are introduced to show situations in which the analysis of such cross-effects is manageable.

Comments on Problems

6.1 Another use of the Cobb-Douglas utility function that shows that cross-price effects are zero. Explaining why they are zero helps to illustrate the substitution and income effects that arise in such situations.

6.2 Shows how some information about cross-price effects can be derived from studying budget constraints alone. In this case, Giffen's Paradox implies that spending on all other goods must decline when the price of a Giffen good rises.

6.3 A simple case of how goods consumed in fixed proportion can be treated as a single commodity (buttered toast).

6.4 An illustration of the composite commodity theorem. Use of the Cobb-Douglas utility produces quite simple results.

6.5 An examination of how the composite commodity theorem can be used to study the effects of transportation or other transactions charges. The analysis here is fairly intuitive—for more detail consult the Borcherding-Silverberg reference.

6.6 Illustrations of some of the applications of the results of Problem 6.5

6.7 This problem demonstrates a special case in which uncompensated cross-price effects are symmetric.

6.8 This problem provides a brief analysis of welfare effects of multiple price changes.

6.9 This is an illustration of the constraints on behavior that are imposed by assuming separability of utility.

6.10 This problem looks at cross-substitution effects in a three good CES function.

Solutions

6.1 a. As for all Cobb-Douglas applications, first-order conditions show that $p_m m = p_s s = 0.5I$. Hence $s = 0.5I/p_s$ and $\partial s/\partial p_m = 0$.

b. Because indifference curves are rectangular hyperboles (*ms* = constant), own substitution and cross-substitution effects are of the same proportional size, but in opposite directions. Because indifference curves are homothetic, income elasticities are 1.0 for both goods, so income effects are also of same proportionate size. Hence, substitution and income effects of changes in p_m on s are precisely balanced.

c.

$$\frac{\partial s}{\partial p_m} = 0 = \frac{\partial s}{\partial p_m}\Big|_{\bar{U}} - m\frac{\partial s}{\partial I} \quad \text{and}$$

$$\frac{\partial m}{\partial p_s} = 0 = \frac{\partial m}{\partial p_s}\Big|_{\bar{U}} - s\frac{\partial m}{\partial I}$$

But $\frac{\partial s}{\partial p_m}\Big|_{\bar{U}} = \frac{\partial m}{\partial p_s}\Big|_{\bar{U}}$ so $m\frac{\partial s}{\partial I} = s\frac{\partial m}{\partial I}$.

d. From part a: $m\frac{\partial s}{\partial I} = m\left(\frac{0.5}{p_s}\right) = m\left(\frac{0.5}{p_m m/s}\right) = s\left(\frac{0.5}{p_m}\right) = s\frac{\partial m}{\partial I}$.

6.2 Since $\partial r/\partial p_r > 0$, a rise in p_r implies that $p_r \cdot r$ definitely rises. Hence, $p_j j = I - p_r r$ must fall, so j must fall. Hence, $\partial j/\partial p_r < 0$.

6.3 a. Yes, $p_{bt} = 2p_b + p_t$.

b. Since $p_c \cdot c = 0.5I,$ $\partial c/\partial p_{bt} = 0$.

c. Since changes in p_b or p_t affect only p_{bt} , these derivatives are also zero.

6.4 a. Amount spent on ground transportation

$$= p_b b + p_t t = p_b\left(b + \frac{p_t}{p_b}\cdot t\right)$$

$$= p_b \cdot g \quad \text{where} \quad g = b + \frac{p_t}{p_b}\cdot t .$$

b. Maximize $U(b, t, p)$ subject to $p_p p + p_b b + p_t t = I$.
This is equivalent to Max $U(g, p) = g^2 p$
Subject to $p_p p + p_b g = I$.

c. Solution is $p = \dfrac{I}{3p_p} \qquad g = \dfrac{2I}{3p_b}$

d. Given $p_b g$, choose $p_b b = p_b g/2 \qquad p_t t = p_b g/2$.

6.5 a. Composite commodity = $p_2 x_2 + p_3 x_3 = p_3\ (kx_2 + x_3)$

b. Relative price $= \dfrac{p_2 + t}{p_3 + t} = \dfrac{kp_3 + t}{p_3 + t}$

Relative price < 1 for $t = 0$. Approaches 1 as $t \to \infty$. Hence, increases in t raise relative price of x_2.

c. Might think increases in t would reduce expenditures on the composite commodity although theorem does not apply directly because, as part (b) shows, changes in t also change relative prices.

d. Rise in t should reduce relative spending on x_2 more than on x_1 since this raises its relative price (but see Borcherding and Silberberg analysis).

6.6 a. Transport charges make low-quality produce relatively more expensive at distant locations. Hence buyers will have a preference for high quality.

b. Increase in baby-sitting expenses raise the relative price of cheap meals.

c. High-wage individuals have higher value of time and hence a lower relative price of Concorde flights.

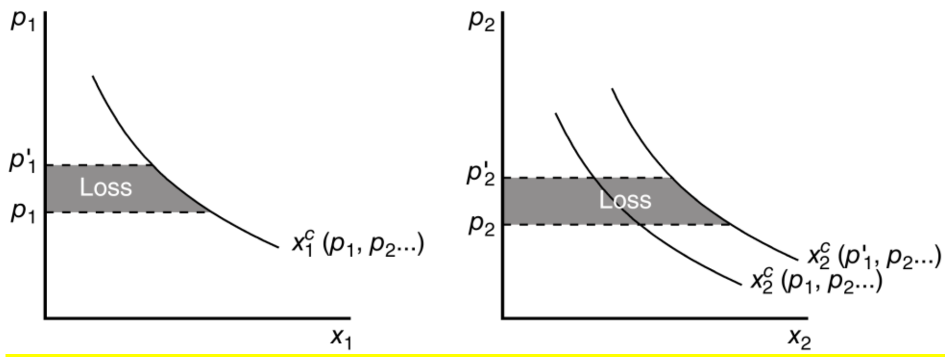
d. Increasing search costs lower the relative price of expensive items.

6.7 Assume $x_i = a_i I \qquad x_j = a_j I$

Hence: $x_j \dfrac{\partial x_i}{\partial I} = a_i a_j I = x_i \dfrac{\partial x_j}{\partial I}$

so income effects (in addition to substitution effects) are symmetric.

6.8 a. $CV = E(p_1', p_2', \overline{p}_3, \ldots \overline{p}_n, \overline{U}) - E(p_1, p_2, \overline{p}_3, \ldots \overline{p}_n, \overline{U})$.

b. 

Notice that the rise p_1 shifts the compensated demand curve for x_2.

c. Symmetry of compensated cross-price effects implies that order of calculation is irrelevant.

d. The figure in part a suggests that compensation should be smaller for net complements than for net substitutes.

6.9 a. This functional form assumes $U_{xy} = 0$. That is, the marginal utility of x does not depend on the amount of y consumed. Though unlikely in a strict sense, this independence might hold for large consumption aggregates such as "food" and "housing."

b. Because utility maximization requires $MU_x/p_x = MU_y/p_y$, an increase in income with no change in p_x or p_y must cause both x and y to increase to maintain this equality (assuming $U_i > 0$ and $U_{ii} < 0$).

c. Again, using $MU_x/p_x = MU_y/p_y$, a rise in p_x will cause x to fall, MU_x to rise. So the direction of change in MU_x/p_x is indeterminate. Hence, the change in y is also indeterminate.

d. If $U = x^\alpha y^\beta \quad MU_x = \alpha x^{\alpha-1} y^\beta$

But $\ln U = \alpha \ln x + \beta \ln y \quad MU_x = \alpha/x$.

Hence, the first case is not separable; the second is.

6.10 a. Example 6.3 gives $x = \dfrac{I}{p_x + \sqrt{p_x p_y} + \sqrt{p_x p_z}}$ clearly $\partial x/\partial p_y, \partial x/\partial p_z < 0$ so these are gross complements.

b. Slutsky Equation shows $\partial x/\partial p_y = \partial x/\partial p_y|_{U=\bar{U}} - y\dfrac{\partial x}{\partial I}$ so $\partial x/\partial p_y|_{U=\bar{U}}$ could be positive or negative. Because of symmetry of y and z here, Hick's second law suggests $\partial x/\partial p_y|_{U=\bar{U}}$ and $\partial x/\partial p_z|_{U=\bar{U}} > 0$.

CHAPTER 7

PRODUCTION FUNCTIONS

Because the problems in this chapter do not involve optimization (cost minimization principles are not presented until Chapter 8), they tend to have a rather uninteresting focus on functional form. Computation of marginal and average productivity functions is stressed along with a few applications of Euler's theorem. Instructors may want to assign one or two of these problems for practice with specific functions, but the focus for Part 3 problems should probably be on those in Chapters 8 and 9.

Comments on Problems

7.1 This problem illustrates the isoquant map for fixed proportions production functions. Parts (c) and (d) show how variable proportions situations might be viewed as limiting cases of a number of fixed proportions technologies.

7.2 This provides some practice with graphing isoquants and marginal productivity relationships.

7.3 This problem explores a specific Cobb-Douglas case and begins to introduce some ideas about cost minimization and its relationship to marginal productivities.

7.4 This is a theoretical problem that explores the concept of "local returns to scale." The final part to the problem illustrates a rather synthetic production that exhibits variable returns to scale.

7.5 This is a thorough examination of all of the properties of the general two-input Cobb-Douglas production function.

7.6 This problem is an examination of the marginal productivity relations for the CES production function.

7.7 This illustrates a generalized Leontief production function. Provides a two-input illustration of the general case, which is treated in the extensions.

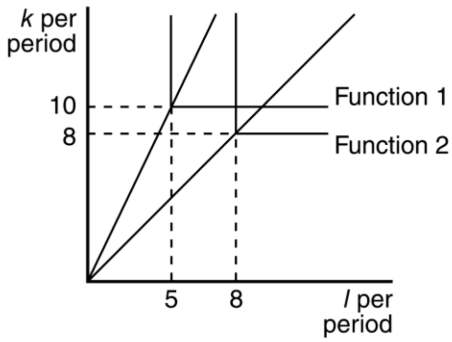
7.8 Application of Euler's theorem to analyze what are sometimes termed the "stages" of the average-marginal productivity relationship. The terms "extensive" and "intensive" margin of production might also be introduced here, although that usage appears to be archaic.

7.9 Another simple application of Euler's theorem that shows in some cases cross second-order partials in production functions may have determinable signs.

7.10 This is an examination of the functional definition of the elasticity of substitution. It shows that the definition can be applied to non-constant returns to scale function if returns to scale takes a simple form.

Solutions

7.1 a., b.



function 1: use $10k$, $5l$

function 2: use $8k$, $8l$

c. Function 1: $2k + l = 8{,}000$

$$2.5(2k + l) = 20{,}000$$

$$5.0k + 2.5l = 20{,}000$$

Function 2: $k + l = 5{,}000$

$$4(k + l) = 20{,}000$$

$$4k + 4l = 20{,}000$$

Thus, $9.0k$, $6.5l$ is on the 40,000 isoquant

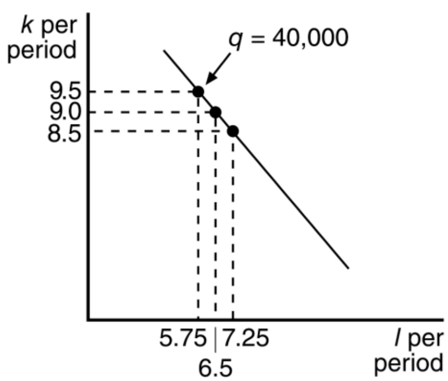
Function 1: $3.75(2k + l) = 30{,}000$

$$7.50k + 3.75l = 30{,}000$$

Function 2: $2(k + l) = 10{,}000$

$$2k + 2l = 10{,}000$$

Thus, $9.5k$, $5.75l$ is on the 40,000 isoquant



7.2 $q = kl - 0.8k^2 - 0.2l^2$

a. When k = 10, $10l - 0.2l^2 - 80\, q = 10L - 80 - .2L^2$.

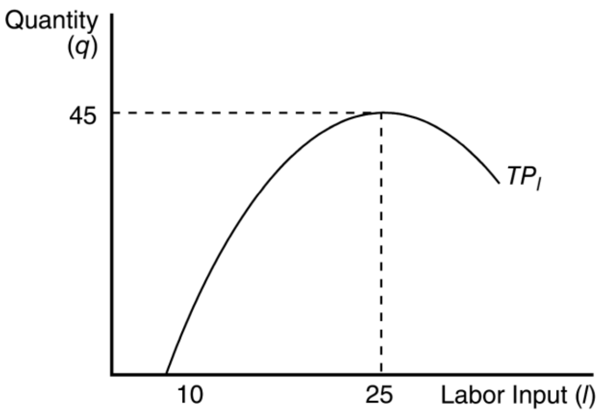
Marginal productivity = $\frac{dq}{dl} = 10 - .4l = 0$, maximum at l = 25

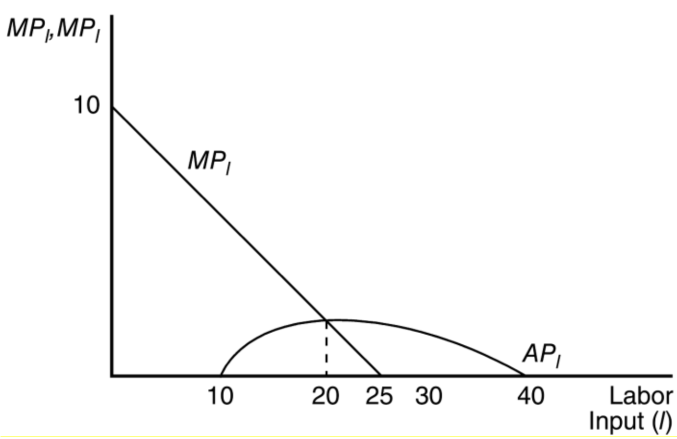
$\frac{d^2q}{dl^2} = -.4$, The total product curve is concave.

$AP_l = q / l = 10 - .2l - 80 / l$

To graph this curve: $\frac{dAP_l}{dl} = \frac{80}{l} - .2 = 0$, maximum at l = 20.

When l = 20, q = 40, AP_l = 0 where l = 10, 40.





b. $MP_l = 10 - .4l$, $10 - .4l = 0$, $l = 25$

See above graph.

c. $k = 20$ $q = 20l - .2l^2 - 320$

$AP_l = 20 - .2l - \frac{320}{l}$; reaches max. at l = 40, $q = 160$

$MPl = 20 - .4l$, $=$ 0 at $\mathrm{l} = 50$.

d. Doubling of k and l here multiplies output by 4 (compare a and c). Hence the function exhibits increasing returns to scale.

7.3 $q = 0.1k^{0.2}l^{0.8}$

a. q = 10 if k = 100, l = 100. Total cost is 10,000.

b. $MP_k = \partial q/\partial k = .02(l/k)^{0.8}, MP_l = .08(k/l)^{0.2}$. Setting these equal yields $l/k = 4$.
Solving $q = 10 = 0.1k^{0.2}(4k)^{0.8} = 0.303k$. So k = 3.3, l = 13.2.
Total cost is 8,250.

c. Because the production function is constant returns to scale, just increase all inputs and output by the ratio 10,000/8250 = 1.21. Hence, k = 4, l = 16, q = 12.1.

d. Carla's ability to influence the decision depends on whether she provides a unique input for Cheers.

7.4 a. If

$$f(tk,tl) = tf(k,l),$$

$$e_{q,t} = \lim(t \to 1)\frac{\partial f(tk,tl)}{\partial t} \cdot \frac{t}{f(tk,tl)} = \lim(t \to 1)\frac{f(k,l)}{f(k,l)} = 1.$$

b. $$e_{q,t} = \lim(t \to 1)\frac{\partial f(tk,tl)}{\partial t} \cdot \frac{t}{f(tk,tl)} = \lim(t \to 1)\left(\frac{\partial f}{\partial k} \cdot k + \frac{\partial f}{\partial l} \cdot l\right) \cdot \frac{t}{f} = e_{q,k} + e_{q,l}$$

c.

$$e_{q,t} = \lim \frac{\partial(1 + t^{-2}k^{-1}l^{-1})}{\partial t} \cdot \frac{t}{q} = \lim q^2 2t^{-3}k^{-1}l^{-1} \cdot \frac{t}{q} = 2qk^{-1}l^{-1}$$

$$= 2q(\frac{1}{q} - 1) = 2 - 2q$$

Hence, $e_{q,t} > 1$ for $q < 0.5$, $e_{q,t} < 1$ for $q > 0.5$.

d. The intuitive reason for the changing scale elasticity is that this function has an upper bound of q = 1 and gains from increasing the inputs decline as q approaches this bound.

7.5 $q = Ak^{\alpha}l^{\beta}$

a.

$$f_k = \alpha Ak^{\alpha-1}l^{\beta} > 0$$
$$f_l = \beta Ak^{\alpha}l^{\beta-1} > 0$$
$$f_{kk} = \alpha(\alpha-1)Ak^{\alpha-2}l^{\beta} < 0$$
$$f_{ll} = \beta(\beta-1)Ak^{\alpha}l^{\beta-2} < 0$$
$$f_{kl} = \alpha\beta Ak^{\alpha-1}l^{\beta-1} > 0$$

b.

$$e_{q,k} = \frac{\partial q}{\partial k}\cdot\frac{k}{q} = \alpha Ak^{\alpha-1}l^{\beta}\cdot\frac{k}{q} = \alpha$$
$$e_{q,l} = \frac{\partial q}{\partial l}\cdot\frac{l}{q} = \beta Ak^{\alpha}l^{\beta-1}\cdot\frac{k}{q} = \beta.$$

c.

$$f(tk,tl) = t^{\alpha+\beta}Ak^{\alpha}l^{\beta}$$
$$e_{q,t} = \lim(t \to 1)\frac{\partial q}{\partial t}\cdot\frac{t}{q} = \lim(\alpha+\beta)t^{\alpha+\beta-1}q\cdot\frac{t}{q} = \alpha+\beta$$

d. Quasiconcavity follows from the signs in part a.

e. Concavity looks at:

$$f_{kk}f_{ll} - f_{kl}^2 = \alpha(\alpha-1)\beta(\beta-1)A^2k^{2\alpha-2}l^{2\beta-2} - \alpha^2\beta^2A^2k^{2\alpha-2}l^{2\beta-2}$$
$$= A^2k^{2\alpha-2}l^{2\beta-2}\alpha\beta(1-\alpha-\beta)$$

This expression is positive (and the function is concave) only if $\alpha+\beta<1$

7.6 a. $MP_k = \dfrac{\partial q}{\partial k} = \dfrac{1}{\rho}\left[k^{\rho}+l^{\rho}\right]^{(1-\rho)/\rho}\cdot \rho k^{\rho-1} = q^{1-\rho}\cdot k^{\rho-1} = (q/k)^{1-\rho}$

Similar manipulations yield $MP_l = \left(\dfrac{q}{l}\right)^{1-\rho}$

b. $RTS = MP_k / MP_l = (l/k)^{1-\rho}$

c. $e_{q,k} = \partial q/\partial k \cdot k/q = (q/k)^{-\rho} = \dfrac{1}{1+(l/k)^{\rho}}$

$$e_{q,l} = (q/l)^{-\rho} = \frac{1}{1+(k/l)^{\rho}} = \frac{1}{1+(l/k)^{-\rho}}$$

Putting these over a common denominator yields $e_{q,k} + e_{q,l} = 1$ which shows constant returns to scale.

d. Since σ = $\dfrac{1}{1-\rho}$ the result follows directly from part a.

7.7 a. If $q = \beta_0 + \beta_1 \sqrt{kl} + \beta_2 k + \beta_3 l$ doubling *k*, *l* gives
$q' = \beta_0 + 2\beta_1 \sqrt{kl} + 2\beta_2 k + 2\beta_3 l = 2q$ when $\beta_0 = 0$

b. $MP_l = 0.5\beta_1 (k/l)^{0.5} + \beta_3 \quad MP_k = 0.5\beta_1 (l/k)^{0.5} + \beta_2$ which are homogeneous of degree zero with respect to *k* and *l* and exhibit diminishing marginal productivities.

c. $$\sigma = \frac{(\partial q/\partial l)\cdot(\partial q/\partial k)}{q\cdot\dfrac{\partial^2 q}{\partial k\,\partial l}}$$

$$= \beta_1^2 + \frac{\beta_1[\beta_2(k/l)^{.5} + \beta_3(l/k)^{.5}] + \beta_2\beta_3}{q\,[0.25\beta_1(kl)^{-.5}]}$$ which clearly varies for different values of *k*, *l*.

7.8 $q = f(k,l)$ exhibits constant returns to scale. Thus, for any *t* > 0, $f(tk,tl) = tf(k,l)$.

Euler's theorem states $tf(k,l) = f_k k + f_l l$. Here we apply the theorem for the case where t = 1: hence, $q = f(k,l) = f_k k + f_l l, \quad q/l = f_l + f_k(k/l)$. If $f_l > q/l$ then $f_k < 0$, hence no firm would ever produce in such a range.

7.9 If $q = f_k k + f_l l$, partial differentiation by *l* yields $f_l = f_{kl} k + f_{ll} l + f_l$. Because $f_{ll} < 0$, $f_{kl} > 0$. That is, with only two inputs and constant returns to scale, an increase in one input must increase the marginal productivity of the other input.

7.10 a. This transformation does not affect the *RTS*:

$$RTS = \frac{F_l}{F_k} = \frac{\gamma f^{\gamma-1} f_l}{\gamma f^{\gamma-1} f_k} = \frac{f_l}{f_k}.$$ Hence, by definition, the value of σ is the same for both functions. The mathematical proof is burdensome, however.

b. The *RTS* for the *CES* function is $RTS = (l/k)^{1-\rho} = (l/k)^{1/\sigma}$. This is not affected by the power transformation.

CHAPTER 8

COST FUNCTIONS

The problems in this chapter focus mainly on the relationship between production and cost functions. Most of the examples developed are based on the Cobb-Douglas function (or its CES generalization) although a few of the easier ones employ a fixed proportions assumption. Two of the problems (8.9 and 8.10) make some use of Shephard's Lemma since it is in describing the relationship between cost functions and (contingent) input demand that this envelope-type result is most often encountered.

Comments on Problems

8.1 Famous example of Viner's draftsman. This may be used for historical interest or as a way of stressing the tangencies inherent in envelope relationships .

8.2 An introduction to the concept of "economies of scope." This problem illustrates the connection between that concept and the notion of increasing returns to scale.

8.3 A simplified numerical Cobb-Douglas example in which one of the inputs is held fixed.

8.4 A fixed proportion example. The very easy algebra in this problem may help to solidify basic concepts.

8.5 This problem derives cost concepts for the Cobb-Douglas production function with one fixed input. Most of the calculations are very simple. Later parts of the problem illustrate the envelope notion with cost curves.

8.6 Another example based on the Cobb-Douglas with fixed capital. Shows that in order to minimize costs, marginal costs must be equal at each production facility. Might discuss how this principle is applied in practice by, say, electric companies with multiple generating facilities.

8.7 This problem focuses on the *CES* cost function. It illustrates how input shares behave in response to changes in input prices and thereby generalizes the fixed share result for the Cobb-Douglas.

8.8 This problem introduces elasticity concepts associated with contingent input demand. Many of these are quite similar to those introduced in demand theory.

8.9 Shows students that the process of deriving cost functions from production functions can be reversed. Might point out, therefore, that parameters of the production function (returns to scale, elasticity of substitution, factor shares) can be derived from cost functions as well—if that is more convenient.

8.10 Illustrates a cost function that arises from a very simple CES production function.

Solutions

8.1 Support the draftsman. It's geometrically obvious that *SAC* cannot be at minimum because it is tangent to *AC* at a point with a negative slope. The only tangency occurs at minimum *AC*.

8.2 a. By definition total costs are lower when both q_1 and q_2 are produced by the same firm than when the same output levels are produced by different firms [C(q_1,0) simply means that a firm produces only q_1].

b. Let $q = q_1+q_2$, where both q_1 and q_2 >0. Because $C(q_1,q_2)/q < C(q_1,0)/q_1$ by assumption, $q_1C(q_1,q_2)/q < C(q_1,0)$. Similarly $q_2C(q_1,q_2)/q < C(0,q_2)$. Summing yields $C(q_1,q_2) < C(q_1,0) + C(0,q_2)$, which proves economies of scope.

8.3 a. $q = 900^{0.5}J^{0.5} = 30\sqrt{J}$ $J = 25 \quad q = 150$

$J = 100 \quad q = 300$

$J = 225 \quad q = 450$

b. Cost = 12 J = $12q^2/900$

$$MC = \frac{dC}{dq} = \frac{24q}{900} = \frac{2q}{75}$$

$q = 150 \quad MC = 4$

$q = 300 \quad MC = 8$

$q = 450 \quad MC = 12$

8.4 $q = \min(5k, 10l) \quad v = 1 \quad w = 3 \quad C = vk + wl = k + 3l$

a. In the long run, keep $5k = 10$, $k = 2l$

$$C = 2l + 3l = 5l = 0.5q \qquad AC = \frac{5l}{10l} = 0.5 \qquad MC = 0.5.$$

b. $k = 10 \quad q = \min(50, 10l)$

$l < 5, q = 10l \qquad C = 10 + 3l = 10 + 0.3q$

$$AC = \frac{10}{q} + 0.3$$

If $l > 5$, $q = 50$ $C = 10 + 3l$ $AC = \frac{10+3l}{50}$

MC is infinite for $q > 50$.

$MC_{10} = MC_{50} = .3$.

MC_{100} is infinite.

8.5 a. $q = 2\sqrt{kl}$, $k = 100, q = 2\sqrt{100\,l}$ $q = 20\sqrt{l}$

$$\sqrt{l} = \frac{q}{20} \quad l = \frac{q^2}{400}$$

$$SC = vK + wL = 1(100) + 4\left(\frac{q^2}{400}\right) = 100 + \frac{q^2}{100}$$

$$SAC = \frac{SC}{q} = \frac{100}{q} + \frac{q}{100}$$

b. $SMC = \frac{q}{50}$. *If* $q = 25, SC = 100 + \left(\frac{25^2}{100}\right) = 106.25$

$$SAC = \frac{100}{25} + \frac{25}{100} = 4.25 \quad SMC = \frac{25}{50} = .50$$

If $q = 50$, $SC = 100 + \left(\frac{50^2}{100}\right) = 125$

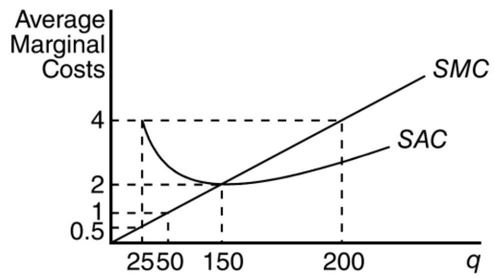
$$SAC = \frac{100}{50} + \frac{50}{100} = 2.50 \quad SMC = \frac{50}{50} = 1$$

If $q = 100$, $SC = 100 + \left(\frac{100^2}{100}\right) = 200$

$$SAC = \frac{100}{100} + \frac{100}{100} = 2 \quad SMC = \frac{100}{50} = 2.$$

If $q = 200$, $SC = 100 + \left(\frac{200^2}{100}\right) = 500$

$$SAC = \frac{100}{200} + \frac{200}{100} = 2.50 \quad SMC = \frac{200}{50} = 4.$$

c. 

d. As long as the marginal cost of producing one more unit is below the average-cost curve, average costs will be falling. Similarly, if the marginal cost of producing one more unit is higher than the average cost, then average costs will be rising. Therefore, the *SMC* curve must intersect the *SAC* curve at its lowest point.

e. $q = 2\sqrt{k\bar{l}}\ so\ q^2 = 4\bar{k}l \quad l = q^2 / 4\bar{k}$

$SC = v\bar{k} + wl = v\bar{k} + wq^2 / 4\bar{k}$

f. $\frac{\partial SC}{\partial k} = v - wq^2 / 4\bar{k}^{-2} = 0 \text{ so } \bar{k} = 0.5qw^{0.5}v^{-0.5}$

g. $C = vk + wl = 0.5q\,w^{0.5}v^{0.5} + 0.5q\,w^{0.5}v^{0.5} = qw^{0.5}v^{0.5}$ (a special case of Example 8.2)

h. If $w = 4 \quad v = 1, \quad C = 2q$

$SC = (\bar{k} = 100) = 100 + q^2/100$, $SC = 200 = C$ for $q = 100$

$SC = (\bar{k} = 200) = 200 + q^2/200$, $SC = 400 = C$ for $q = 200$

$SC = 800 = C$ for $q = 400$

8.6 a. $q_{\text{total}} = q_1 + q_2$. $q_1 = \sqrt{25l_1} = 5\sqrt{l_1} \quad q_2 = 10\sqrt{l_2}$

$SC_1 = 25 + l_1 = 25 + q_1^2/25 \quad SC_2 = 100 + q_2^2/100$

$SC_{\text{total}} = SC_1 + SC_2 = 125 + \frac{q_1^2}{25} + \frac{q_2^2}{100}$

To minimize cost, set up Lagrangian: $? = SC + \lambda\ q - q_1 - q_2)$.

$$\frac{\partial £}{\partial q_1} = \frac{2q_1}{25} - \lambda = 0$$

$$\frac{\partial £}{\partial q_2} = \frac{2q_2}{100} - \lambda = 0$$

Therefore $q_1 = 0.25q_2$.

b. $4q_1 = q_2 \quad q_1 = 1/5\,q \quad q_2 = 4/5\,q$

$$SC = 125 + \frac{q^2}{125} \quad SMC = \frac{2q}{125} \quad SAC = \frac{125}{q} + \frac{q}{125}$$

$$SMC\ (100) = \frac{200}{125} = \$1.60$$

$SMC(125) = \$2.00$ $SMC(200) = \$3.20$

c. In the long run, can change *k* so, given constant returns to scale, location doesn't really matter. Could split evenly or produce all output in one location, etc.

$C = k + l = 2q$

$AC = 2 = MC$

d. If there are decreasing returns to scale with identical production functions, then should let each firm have equal share of production. *AC* and *MC* not constant anymore, becoming increasing functions of *q*.

8.7 a. $C = q^{1/\gamma}[(v/a)^{1-\sigma} + (w/b)^{1-\sigma}]^{1/1-\sigma}$.

b. $C = qa^{-a}b^{-b}v^a w^b$.

c. $wl/vk = b/a$.

d. $l/k = [\frac{(v/a)}{(w/b)}]^{\sigma}$ so $wl/vk = (v/w)^{\sigma-1}(b/a)^{\sigma}$. Labor's relative share is an increasing function of *b/a*. If σ > 1 labor's share moves in the same direction as *v/w*. If σ < 1, labor's relative share moves in the opposite direction to *v/w*. This accords with intuition on how substitutability should affect shares.

8.8 a. The elasticities can be read directly from the contingent demand functions in Example 8.4. For the fixed proportions case, $e_{l^c,w} = e_{k^c,v} = 0$ (because *q* is held constant). For the Cobb-Douglas, $e_{l^c,w} = -\alpha/\alpha + \beta, e_{k^c,v} = -\beta/\alpha + \beta$. Apparently the *CES* in this form has non-constant elasticities.

b. Because cost functions are homogeneous of degree one in input prices, contingent demand functions are homogeneous of degree zero in those prices as intuition suggests. Using Euler's theorem gives $l_w^c w + l_v^c v = 0$. Dividing by l^c gives the result.

c. Use Young's Theorem:

$$\frac{\partial l^c}{\partial v} = \frac{\partial^2 C}{\partial v \partial w} = \frac{\partial^2 C}{\partial w \partial v} = \frac{\partial k^c}{\partial w}$$ Now multiply left by $\frac{vwl^c}{l^c C}$ right by $\frac{vwk^c}{k^c C}$.

d. Multiplying by shares in part b yields $s_l e_{l^c,w} + s_l e_{l^c,v} = 0$. Substituting from part c yields $s_l e_{l^c,w} + s_k e_{k^c,w} = 0$.

e. All of these results give important checks to be used in empirical work.

8.9 From Shephard's Lemma

a. $l = \frac{\partial C}{\partial w} = \frac{2}{3} q \left(\frac{v}{w} \right)^{1/3} \quad k = \frac{\partial C}{\partial v} = \frac{1}{3} q \left(\frac{w}{v} \right)^{2/3}$

b. Eliminating the *w/v* from these equations:

$q = \left(\frac{3}{2} \right)^{2/3} (3)^{1/3} l^{2/3} k^{1/3} = B l^{2/3} k^{1/3}$ which is a Cobb-Douglas production function.

8.10 As for many proofs involving duality, this one can be algebraically messy unless one sees the trick. Here the trick is to let B = ($v^{.5} + w^{.5}$). With this notation, $C = B^2 q$.

a. Using Shephard's lemma,

$k = \frac{\partial C}{\partial v} = B v^{-0.5} q \quad l = \frac{\partial C}{\partial w} = B w^{-0.5} q.$

b. From part a,

$\frac{q}{k} = \frac{v^{0.5}}{B}, \quad \frac{q}{l} = \frac{w^{0.5}}{B} \quad so \quad \frac{q}{k} + \frac{q}{l} = 1 \quad or \quad k^{-1} + l^{-1} = q^{-1}$

The production function then is $q = (k^{-1} + l^{-1})^{-1}$.

c. This is a CES production function with ρ = –1. Hence, σ = 1/(1 – ρ) = 0.5. Comparison to Example 8.2 shows the relationship between the parameters of the *CES* production function and its related cost function.

CHAPTER 9

PROFIT MAXIMIZATION

Problems in this chapter consist mainly of applications of the $P = MC$ rule for profit maximization by a price-taking firm. A few of the problems (9.2–9.5) ask students to derive marginal revenue concepts, but this concept is not really used in the monopoly context until Chapter 13. The problems are also concerned only with the construction of supply curves and related concepts since the details of price determination have not yet been developed in the text.

Comments on Problems

9.1 A very simple application of the $P = MC$ rule. Results in a linear supply curve.

9.2 Easy problem that shows that a tax on profits will not affect the profit-maximization output choice unless it affects the relationship between marginal revenue and marginal cost.

9.3 Practice with calculating the marginal revenue curve for a variety of demand curves.

9.4 Uses the *MR*-*MC* condition to illustrate third degree price discrimination. Instructors might point out the general result here (which is discussed more fully in Chapter 13) that, assuming marginal costs are the same in the two markets, marginal revenues should also be equal and that implies price will be higher in the market in which demand is less elastic.

9.5 An algebraic example of the supply function concept. This is a good illustration of why supply curves are in reality only two-dimensional representations of multi-variable functions.

9.6 An introduction to the theory of supply under uncertainty. This example shows that setting expected price equal to marginal cost does indeed maximize expected revenues, but that, for risk-averse firms, this may not maximize expected utility. Part (d) asks students to calculate the value of better information.

9.7 A simple use of the profit function with fixed proportions technology.

9.8 This is a conceptual examination of the effect of changes in output price on input demand.

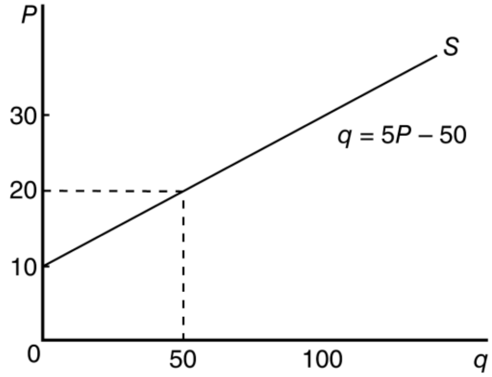
9.9 A very brief introduction to the *CES* profit function.

9.10 This problem describes some additional mathematical relationships that can be derived from the profit function.

Solutions

9.1 a. $MC = \partial C/\partial q = 0.2q + 10$ set $MC = P = 20$, yields $q^* = 50$

b. $\pi = Pq - C = 1000 - 800 = 200$

c. 

9.2 $\pi(q) = R(q) - C(q)$ With a lump sum tax T

$\pi(q) = R(q) - C(q) - T$ $\frac{\partial \pi}{\partial q} = \frac{\partial R}{\partial q} - \frac{\partial C}{\partial q} - 0 = 0$ $MR = MC$, no change

Proportional tax $\pi(q) = (1-t)[R(q) - C(q)]$

$\frac{\partial \pi}{\partial q} = (1-t)(MR - MC) = 0,$ $MR = MC$, no change

Tax per unit $\pi(q) = R(q) - C(q) - tq$

$\frac{\partial \pi}{\partial q} = MR - MC - t = 0$, so $MR = MC + t$, q is changed: a per unit tax does affect output.

9.3 a. $q = a + bP$, $MR = P + q\frac{dP}{dq} = \frac{q-a}{b} + q\ (1/b) = \frac{2q-a}{b}$

Hence, $q = (a + bMR)/2$.

Because the distance between the vertical axis and the demand curve is $q = a + bP$, it is obvious that the marginal revenue curve bisects this distance for any line parallel to the horizontal axis.

b. If $q = a + bP$; $b < 0$; $P = \frac{q-a}{b}$

$MR = \frac{2q-a}{b}$ $P - MR = -\frac{1}{b}q$

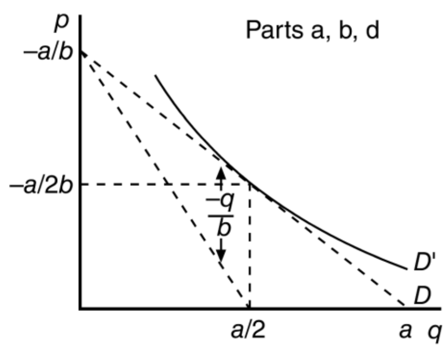
c. Constant elasticity demand curve: $q = aP^b$, where b is the price elasticity of demand.

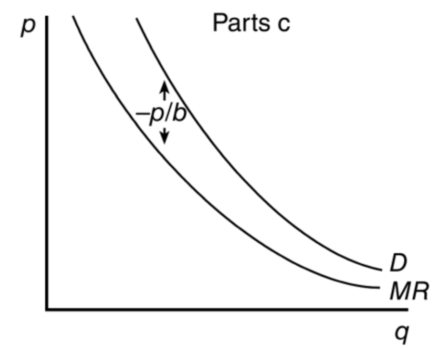
$$MR = P + q\frac{\partial P}{\partial q} = \left(\frac{q}{a}\right)^{1/b} + \left(\frac{(q/a)^{1/b}}{b}\right)$$

Thus, vertical distance = $P - MR = \frac{-(q/a)^{1/b}}{b} = \frac{-P}{b}$ (which is positive because $b < 0$)

d. If $e_{q,P} < 0$ (downward-sloping demand curve), then marginal revenue will be less than price. Hence, vertical distance will be given by $P - MR$.

Since $MR = P + q\frac{dP}{dq}$, vertical distance is $-q\frac{dP}{dq}$, and since $\frac{dq}{dP} = b$ is the slope of the tangent linear demand curve, the distance becomes $-\frac{1}{b}q$ as in Part (b).





e.

9.4 Total cost = $C = .25q^2 = .25(q_A + q_L)^2$

$q_A = 100 - 2P_A$ $\quad$ $q_L = 100 - 4P_L$

$P_A = 50 - q_A/2$ $\quad$ $P_L = 25 - q_L/4$

$R_A = P_A q_A = 50q_A - q_A^2/2$ $\quad$ $R_L = P_L q_L = 25q_L - q_L^2/4$

$MR_A = 50 - q_A$ $\quad$ $MR_L = 25 - q_L/2$

$MC_A = .5(q_A + q_L)$ $\quad$ $MC_L = .5(q_A + q_L)$

Set $MR_A = MC_A$ and $MR_L = MC_L$

$50 - q_A = .5q_A + .5q_L \qquad 25 - \frac{q_L}{2} = .5q_A + .5q_L$

Solving these simultaneously gives

$q_A = 30 \qquad P_A = 35$

$q_L = 10 \qquad P_L = 22.5 \qquad \pi = 1050 + 225 - 400 = 875$

9.5 a. Since $q = 2\sqrt{l}$, $q^2 = 4l \quad C = wl = wq^2/4$.

Profit maximization requires $P = MC = 2wq/4$.

Solving for q yields $q = 2P/w$.

b. Doubling P and w does not change profit-maximizing output level.

$\pi = Pq - TC = 2P^2/w - P^2/w = P^2/w$, which is homogeneous of degree one in P and w.

c. It is algebraically obvious that increases in w reduce quantity supplied at each given P.

9.6 a. Expected profits = $E(\pi) = .5[30q - C(q)] + .5[20q - C(q)] = 25q - C(q)$.

Notice $25 = E(P)$ determines expected profits.

For profit maximum set $E(P) = MC = q + 5$ so $q = 20$

$E(\pi) = E(P)q - C(q) = 500 - 400 = 100$.

b. In the two states of the world profits are

$P = 30 \qquad \pi = 600 - 400 = 200$

$P = 20 \qquad \pi = 400 - 400 = 0$ and expected utility is given by $E(U) = .5\sqrt{200} + .5\sqrt{0} = 7.1$

c. Output levels between 13 and 19 all yield greater utility than does $q = 20$. Reductions in profits from producing less when P is high are compensated for (in utility terms) by increases in profits when P is low. Calculating true maximum expected utility is difficult—it is approximately $q = 17$.

d. If can predict P, set $P = MC$ in each state of the world.

When $P = 30$ $q = 25$ $\pi = 212.5$, $P = 20$ $q = 15$ $\pi = 12.5$
$E(\pi) = 112.5$
$E(U) = .5\sqrt{212.5} + .5\sqrt{12.5} = 9.06$—a substantial improvement.

9.7 a. In order for the second order condition for profit maximization to be satisfied, marginal cost must be decreasing which, in this case, requires diminishing returns to scale.

b.

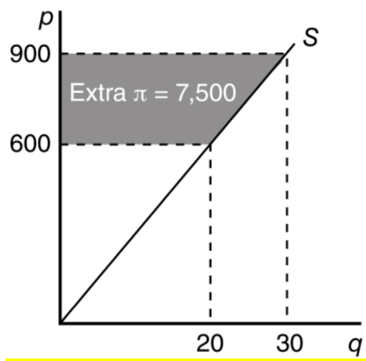
$$q = 10k^{0.5} = 10l^{0.5} \text{ so } k = l = q^2/100$$
$$C = vk + wl = q^2(v+w)/100$$

Profit maximization requires $P = MC = q(v+w)/50$ or $q = 50P/(v+w)$.

$$\Pi(v,w,P) = Pq - C = 50P^2/(v+w) - [50P/(v+w)]^2(v+w)/100 = 25P^2/(v+w).$$

c. If $v = 1000, w = 500, P = 600$ then $q = 20, \pi = 6000$.

d. If $v = 1000, w = 500, P = 900$ then $q = 30, \pi = 13500$.

e. 

9.8 a. With marginal cost increasing, an increase in *P* will be met by an increase in *q*. To produce this extra output, more of each input will be hired (unless an input is inferior).

b. The Cobb-Douglas case is best illustrated in two of the examples in Chapter 9. In Example 9.4, the short-run profit function exhibits a positive effect of *P* on labor demand. A similar result holds in Example 9.5 where holding a third input constant leads to increasing marginal cost.

c. $\partial l/\partial P = \partial[-\partial\Pi/\partial w]/\partial P = -\partial^2\Pi/\partial P\partial w = \partial q/\partial w$. The sign of the final derivative may be negative if l is an inferior input.

9.9 b. Diminishing returns is required if *MC* is to be increasing—the required second order condition for profit maximization.

c. σ determines how easily firms can adapt to differing input prices and thereby shows the profitability obtainable from a given set of exogenous prices.

d. $q = \partial\Pi/\partial P = K(1-\gamma)^{-1}P^{\gamma/1-\gamma}(v^{1-\sigma} + w^{1-\sigma})^{\gamma/(1-\sigma)(\gamma-1)}$. This supply function shows that σ does not affect the supply elasticity directly, but it does affect the shift term that involves input prices. Larger values for σ imply smaller shifts in the supply relationship for given changes in input prices.

e. See the results provided in *Sydsaeter, Strom, and Berck.*

9.10 a. $\partial l/\partial v = \partial^2\Pi/\partial v\partial w = \partial^2\Pi/\partial w\partial v = \partial k/\partial w$. This shows that cross price effects in input demand are equal. The result is similar to the equality of compensated cross-price effects in demand theory.

b. The direction of effect depends on whether capital and labor are substitutable or complementary inputs.

c. $\partial q/\partial w = \partial^2\Pi/\partial w\partial P = \partial^2\Pi/\partial P\partial w = -\partial l/\partial P$. This shows that increases in wages have the same effect on reducing output that a fall in the product price has on reducing labor demand. This is, the effects of wages and prices are in some ways symmetrical.

d. Because it seems likely that $\partial l/\partial P > 0$ (see Problem 9.8), we can conclude that $\partial q/\partial w < 0$ —that is, a tax on labor input should reduce output.

CHAPTER 10

THE PARTIAL EQUILIBRIUM COMPETITIVE MODEL

The problems in this chapter focus on competitive supply behavior in both the short and long runs. For short-run analysis, students are usually asked to construct the industry supply curve (by summing firms' marginal cost curves) and then to describe the resulting market equilibrium. The long-run problems (10.5– 10.8), on the other hand, make extensive use of the equilibrium condition P = MC = AC to derive results. In most cases, students are asked to graph their solutions because, I believe, such graphs provide considerable intuition about what is going on.

Comments on Problems

10.1 This problem asks students to constructs a marginal cost function from a cubic cost function and then use this to derive a supply curve and a supply-demand equilibrium. The math is rather easy so this is a good starting problem.

10.2 A problem that illustrates "interaction effects." As industry output expands, the wage for diamond cutters rises, thereby raising costs for all firms.

10.3 This is a simple, though at times tedious, problem that shows that any one firm's output decision has very little effect on market price. That is shown to be especially true when other firms exhibit an elastic supply response in reaction to any price changes induced by any one firm altering its output. That is, any one firm's effect on price is moderated by the induced effect on other firms.

10.4 This is a tax-incidence problem. It shows that the less elastic the supply curve, the greater the share of tax paid this is paid by firms (for a given demand curve). Issues of tax incidence are discussed in much greater detail in Chapter 11.

10.5 This is a simple problem that uses only long-run analysis. Once students recognize that the equilibrium price will always be $3.00 per bushel and the typical firm always produces 1,000 bushels, the calculations are trivial.

10.6 A problem that is similar to 10.5, but now introduces the short-run supply curve to examine differences in supply response over the short and long runs.

10.7 This problem introduces the concept of increasing input costs into long-sun analysis by assuming that entrepreneurial wages are bid up as the industry expands. Solving part (b) can be a bit tricky; perhaps an educated guess is the best way to proceed.

10.8 This exercise looks at an increase in cost that also shifts the low point of the typical firm's AC curve. Here the increase in cost reduces optimal firm size and has the seemingly odd effect of a cost increase leading to a fall in quantity demanded but an increase in the number of firms.

Solutions

10.1 $C(q)=\frac{1}{300}q^3+.2q^2+4q+10 \quad MC(q)=.01q^2+.4q+4$

a. Short run: $P = MC \quad P = .01q^2 + .4q + 4$

$100P = q^2 + 40q + 400 = (q + 20)^2 = 100P,$

$q+20=10\sqrt{P} \quad q=10\sqrt{P}-20$

b. Industry: $Q=100q=1000\sqrt{P}-2000$

c. Demand: $Q=-200P+8000$ set demand = supply

$-200P+8000=1000\sqrt{P}-2000$

$1000\sqrt{P}+200P=10,000$

$5\sqrt{P}+P=50, P=25 \;\; Q=3000$

For each firm $q = 30$, $C = 400$, $AC = 13.3$, $\pi = 351$.

10.2 $C = q^2 + wq \quad MC=2q+w$

a. $w = 10 \quad C = q^2 + 10q$

$MC = 2q + 10 = P \quad q=0.5P-5$

Industry $Q=\sum_{1}^{1000} q=500P-5000$

at 20, $Q = 5000$; at 21, $Q = 5500$

b. Here, $MC = 2q + .002Q$ for profit maximum, set = P

$q=0.5P-0.001Q$

Total $Q=\sum_{1}^{1000} q=500P-Q \quad Q=250P$

$P = 20$, $Q = 5000$ Supply is more steeply sloped in this case where expanded output bids up wages.

$P = 21$, $Q = 5250$

10.3 a. Very short run, $Q_S = (100)(1000) = 100{,}000$. Since there can be no supply response, this Q must be sold for whatever the market will bear:

$160{,}000 - 10{,}000P = 100{,}000. \qquad P^* = 6$

b. For any one firm, quantity supplied by other firms is fixed at 99,900.

Demand curve facing firm is $q = 160{,}000 - 10{,}000P - 99{,}900 = 60{,}100 - 10{,}000P$.

c. If quantity supplied by the firm is zero,

$q_S = 0 = q_D = 60{,}100 - 10{,}000P. \qquad P^* = 6.01$

If quantity supplied by the firm is 200,

$q_S = 200 = q_D = 60{,}100 - 10{,}000P \qquad P^* = 5.99$

d. $e_{Q,P} = -10{,}000 ? \dfrac{6}{100{,}000} = -6$

For single firm: $e_{q,P} = -10{,}000 ? \dfrac{6}{100} = - \ \ 00.$

Demand facing the single firm is "close to" infinitely elastic. Now redo these parts with short-run supply of $q_i = -200 + 50P$

a. $Q_S = 1000q_i = -200{,}000 + 50{,}000P$

Set supply = demand: Resulting equilibrium price is $P^* = 6$

b. For any one firm, find net demand by subtracting supply by other 999 firms.
$q_D = 160{,}000 - 10{,}000P - (-199{,}800 + 49{,}950P) = 359{,}800 - 59{,}950P$

c. If $q_S = 0,\ P^* = \dfrac{359{,}800}{59{,}950} = 6.002$.

If $q_S = 200,\ P^* = \dfrac{359{,}600}{59{,}950} = 5.998$.

d. Elasticity of the industry demand curve remains the same. Demand curve facing the firm is even more elastic than in the fixed supply case:

$e_{q\,p} = -59{,}950 \cdot \dfrac{6}{100} = -3597.$

10.4 a. $Q_D = 100 - 2P \qquad Q_S = 20 + 6P$

At equilibrium, $Q_D = Q_S$.

$100 - 2P = 20 + 6P \qquad P^* = \$10,\ Q^* = 80$

b. Demanders and suppliers are now faced with different prices $P_S = P_D - 4$. Each will make decisions on quantity based on the price that it is faced with:

$Q_D = 100 - 2P_D \qquad Q_S = 20 + 6P_S = 20 + 6(P_D - 4).$

The new equilibrium: $100 - 2P_D = 20 + 6P_D - 24$

$8P_D = 104, \quad P_D = \$13$

$P_S = \$9, Q = 74$

The burden of the tax is shared: demanders pay \$3 more for each frisbee while suppliers receive \$1 less on each sale.

c. $Q_S = 70 + P$

At equilibrium, $Q_D = Q_S$.

$100 - 2P = 70 + P, \quad P^* = \$10, \quad Q^* = 80$

After tax: $Q_D = 100 - 2P_D$

$Q_S = 70 + P_S = 70 + P_S - 4$

$100 - 2P_D = 70 + P_D - 4$

$3P_D = 34, \; P_D = 11.3, \; P_S = 7.3, \; Q = 77.3$

While burden is still shared, in this case suppliers pay relatively more of the tax.

10.5 a. $Q_D = 2{,}600{,}000 - 200{,}000P$
In the long run, $P = \$3$, so $Q_S = Q_D = 2{,}600{,}000 - 200{,}000(3) = 2{,}000{,}000$.

Since $Q_S = 2{,}000{,}000$ bushels, there are $\dfrac{2000000 \text{ bushels}}{1000 \text{ bushels/farm}} = 2000$ farms

b. $Q_S = Q_D = 3{,}200{,}000 - 200{,}000P$

In the short run, $Q_S = 2{,}000{,}000$, so $2{,}000{,}000 = 3{,}200{,}000 - 200{,}000P$

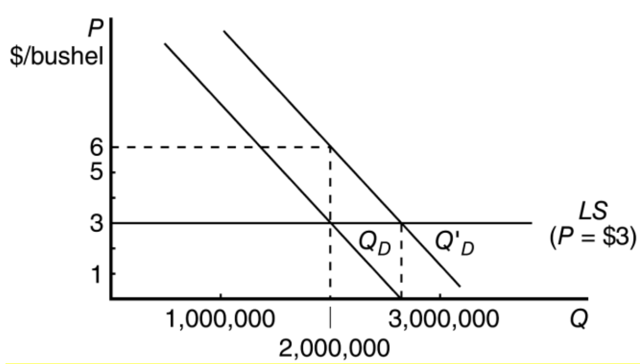
$1{,}200{,}000 = 200{,}000P \qquad P = \$6/\text{bushel}$

$\pi = q(P - AC) = 1000(6 - 3) = 3000$

c. $P = \$3$/bushel in the long run.

$Q_S = Q_D = 3{,}200{,}000 - 200{,}000(3) = 2{,}600{,}000$ bushels

There will be $\dfrac{2{,}600{,}000 \text{ bushels}}{1000 \text{ bushels/farm}} = 2{,}600$ farms.



d.

10.6 a. LR supply curve is horizontal at $P^* = MC = AC = 10$.

b. $Q^* = 1500 - 50P^* = 1000$. Each firm produces

$q^* = 20$, $\pi = 0$. There are 50 firms.

c. $MC = q - 10$, $AC = .5q - 10 + 200/q$

AC = min when $AC = MC$ $.5q = 200/q$, $q = 20$.

d. $P = MC = q - 10$. $q = P + 10$, for industry $Q = \sum_{1}^{50} q = 50P + 500$.

e. $Q = 2000 - 50P$ if $Q = 1000$, $P = 20$. Each firm produces $q = 20$,

$\pi = 20(20 - 10) = 200$.

f. $50P + 500 = 2000 - 50P$ $P = 15$, $Q = 1250$.

Each firm produces $q = 25$, $\pi = 25(15 - AC) = 25(15 - 10.5) = 112.5$.

g. $P^* = 10$ again, $Q = 1500$, 75 firms produce 20 each. $\pi = 0$.

10.7 a. $C(q, w) = 0.5q^2 - 10q + w$ Equilibrium in the entrepreneur market requires $Q_S = 0.25w = Q_D = n$ or $w = 4n$. Hence $c(q, w) = 0.5q^2 - 10q + 4n$.

$MC = q - 10$

$$AC = .5q - 10 + \frac{4n}{q}$$

In long run equilibrium: $AC = MC$ so $q - 10 = .5q - 10 + \frac{4n}{q}$

$$.5q = \frac{4n}{q} \qquad q = \sqrt{8n}$$

Total output is given in terms of the number of firms by $Q = nq = n\sqrt{8n}$. Now in terms of supply-demand equilibrium, $Q_D = 1500 - 50P$ and $P = MC = q - 10$, or $q = P + 10$.

$$Q_S = nq = n(P + 10).$$

Have 3 equations in Q, n, P. Since $Q = n\sqrt{8n}$ and $Q = n(P + 10)$, we have $n\sqrt{8n} = n(P + 10)$ $P = \sqrt{8n} - 10$. $Q_D = 1500 - 50P = 1500 - 50\sqrt{8n} + 500$ $= 2000 - 50\sqrt{8n} = Q_S = n\sqrt{8n}$ or $(n + 50)\sqrt{8n} = 2000$

$n = 50$ (= # of entrepreneurs)

$Q = n\sqrt{8n} = 1000$

$q = Q/n = 20$

$P = q - 10 = 10$

$w = 4n = 200.$

b. Algebra as before, $(n+50)\sqrt{8n} = 2928$, therefore $n = 72$

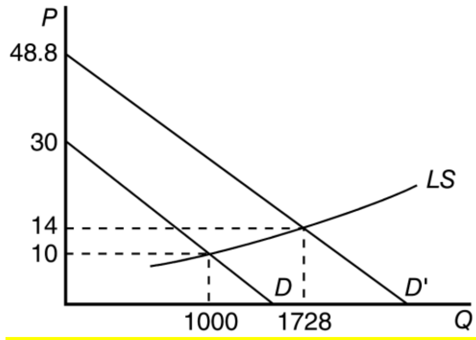
$Q = n\sqrt{8n} = 1728$

$q = Q/n = 24$

$P = q - 10 = 14$

$w = 4n = 288$

c.



This curve is upward sloping because as new firms enter the industry the cost curves shift up: $AC = 0.5q - 10 + (4n/q)$ as n increases, AC increases.

10.8 a. $C(q,w) = wq^2 - 10q + 100$

If $w = \$1$, $C = q^2 - 10q + 100$.

$MC = 2q - 10 \qquad AC = q - 10 + \dfrac{100}{q}$

In the long run equilibrium, $AC = MC$

$2q - 10 = q - 10 + \dfrac{100}{q}$, $q^2 = 100$, $q = 10$ = output for typical mushroom producer.

b. Constant costs industry means that as new firms enter this low point of average, total cost remains unchanged, resulting in a horizontal supply curve at $P = \$10$ (when $q = 10$, $AC = \$10$). Thus, long-run equilibrium $P = \$10$ and $Q = 30{,}000$. There will be $\dfrac{30{,}000}{10} = 3{,}000$ firms.

c. If $w = \$4 \qquad C = 4q^2 - 10q + 100$

$MC = 8q - 10 \qquad AC = 4q - 10 + \dfrac{100}{q}$.

In the long run $\qquad AC = MC$

$8q - 10 = 4q - 10 + \frac{100}{q}$, $q^2 = 25$, $q = 5$.

Long-run equilibrium price = low point of AC, = 30.

Thus, $Q = -1{,}000(30) + 40{,}000 = 10{,}000$.

There will be $\frac{10,000}{5}$ = 2,000 firms.

d. For part (a), still have optimal $q = 10$.

For part (b), now $Q_D = -1{,}000(10) + 60{,}000 = 50{,}000$

so $n = \frac{50,000}{10} = 5,000$

For part (c) $q = 5$, $P = 30$. $Q_D = 30{,}000$

$n = \frac{30,000}{5} = 6,000$.

For part (d) demand is less elastic so reduction in optimal size more than compensates for reduction in quantity demanded as a result of cost increase, so the number of firms rises.

CHAPTER 11

APPLYING THE COMPETITIVE MODEL

The problems in this chapter are intended to illustrate the types of calculations made using simple competitive models for applied welfare analysis. Usually the problems start from a supply-demand framework much like that used for the problems in Chapter 10. Students are then asked to evaluate the effects of changing equilibria on the welfare of market participants. Notice that, throughout the problems, consumer surplus is measured as the area below the Marshallian (uncompensated) demand curve.

Comments on Problems

11.1 Illustrates some simple consumer and producer surplus calculations. Results of this problem are used later to examine price controls (Problem 11.4) and tax incidence (Problem 11.5).

11.2 This problem illustrates the computations of short-run produce surplus in a simple linear case.

11.3 An increasing cost example that illustrates long-run producer surplus. Notice that both producer surplus and rent calculations must be made incrementally so that total values will add-up properly.

11.4 A continuation of Problem 11.1 that examines the welfare consequences of price controls.

11.5 Another continuation of Problem 11.1 that examines tax incidence with a variety of different demand and supply curves. The solutions also provide an elasticity interpretation of this problem.

11.6 A continuation of Problem 11.2 that looks at the effects of taxation on short-run producer surplus.

11.7 A continuation of Problem 11.3 that examines tax incidence, long-run producer surplus, and changes in input rents.

11.8 Provides some simple computations of the deadweight losses involved with tariffs.

11.9 A continuation of Problem 11.8 that examines marginal excess burden. Notice that the increase in the tariff rate actually reduces tariff revenue in this problem.

11.10 A graphical analysis for the case of a country that faces a positively sloped supply curve for imports.

Solutions

11.1 a. Set $Q_D = Q_S$ $1000 - 5P = 4P - 80$ $P^* = 120$, $Q^* = 400$.
For consumers, $P_{\max} = 200$ For producers, $P_{\min} = 20$

$CS = .5(200 - 120)(400) = 16{,}000$

$PS = .5(120 - 20)(400) = 20{,}000$

b. Loss = $.5(100)(P_D - P_S)$ where P_S is the solution to

$300 = 4P_S - 80$ $P_S = 95$

P_D is the solution to $300 = 1000 - 5P_D$ so $P_D = 140$

Total loss of consumer and producer surplus = 50(45) = 2250

c. If $P = 140$

$CS = .5(300)(60) = 9000$

$PS = .5(300)(95 - 20) + 45(300) = 11{,}250 + 13{,}500 = 24{,}750$

Producers gain 4,750, consumers lose 7,000. The difference (2250) is the deadweight loss.

If $P = 95$

$CS = 9000 + 13{,}500 = 22{,}500$

$PS = 11{,}250$

Consumers gain 6500, producers lose 8,750. Difference is again the deadweight loss, 2,250.

d. With $Q = 450$ have

$450 = 1000 - 5P_D$ $P_D = 110$

$450 = 4P_S - 80$ $P_S = 132.5$

Loss of surplus is $0.5 \cdot 50 \cdot (P_S - P_D) = 25(22.5) = 562.5$. As in part c, this total loss is independent of price, which can fall between 110 and 132.5.

11.2 a. Short-run supply is $q = P - 10$, market supply is $100q = 100P - 1000$.

b. Equilibrium where $100P - 1000 = 1100 - 50P$, $P = 14$, $Q = 400$.

c. Since $Q_S = 0$ when $P = 10$, Producer Surplus = $.5(14 - 10)(400) = 800$.

d. Total industry fixed cost = 500.

For a single firm $\pi = 14(4) - [.5(4)^2 + 40 + 5] = 56 - 53 = 3$

Total industry profits = 300

Short-run profits + fixed cost = 800 = producer surplus.

11.3 a. The long-run equilibrium price is $10 + r = 10 + .002Q$.

So, $Q = 1050 - 50(10 + .002Q) = 550 + .1Q$ so

$Q = 500, P = 11, r = 1$.

b. Now $Q = 1600 - 50(10 + .002Q) = 1100 + .1Q$

$Q = 1000, P = 12, r = 2$.

c. Change in $PS = 1(500) + .5(1)(500) = 750$.

d. Change in rents = 1(500) + .5(1)(500) = 750. The areas are equal.

11.4 a. Equilibrium is given by $1270 - 5P = 4P - 80$

$P = 150 \quad Q = 520$.

b. Now the maximum demanders will pay is $P_{\max} = 1270/5 = 254$

$CS = .5(520)(254 - 150) = 27{,}040$

Minimum supply price is $P_{min} = 80/4 = 20$

$PS = .5(520)(150 - 20) = 33{,}800$

c. With P fixed at 120, $Q = 400$

$400 = 1270 - 5P_D$

$P_D = 174$

$CS = .5(400)(254 - 174) + (400)(174 - 120) = 16{,}000 + 21{,}600 = 37{,}600$

$PS = .5(400)(120 - 20) = 20{,}000$.

The change in CS represents a transfer from producers to consumers of 400(150 – 120) = 12,000 less a deadweight loss of .5(120)(174 – 150) = 1440.

The change in PS represents a transfer of 12,000 to consumers and a deadweight loss of .5(120)(150 – 120) = 1,800. Total deadweight loss is 3,240.

11.5 a. Because the gap between P_D and P_S is 45 at $Q = 300$ in Problem 11.1, that is the post tax equilibrium. Total taxes = 13,500.

b. Consumers pay (140 – 120)(300) = 6,000 (46%)

Producers pay (120 – 95)(300) = 7,500 (54%)

c. Excess burden = Deadweight Loss = 2250 from 11.1 part b.

d. $Q_D = 2250 - 15\,P_D = 4P_S - 80 = 4(P_D - 45) - 80$

$19P_D = 2460 \qquad P_D = 129.47 \qquad P_S = 84.47$

$Q = 258$ tax = 11,610

Consumers pay 258(129.47 – 120) = 2,443 (21%)

Producers pay 258(120 – 84.47) = 9,167 (79%)

e. $Q_D = 1000 – 5P_D = 10(P_D – 45) – 800$

$2250 = 15P_D$ $P_D = 150$ $P_S = 105$

$Q = 250$ Total tax = 11,250

Consumers pay 250(150 – 120) = 7500 (67%)

Producers pay 250(120 – 105) = 3750 (33%)

f. Elasticities in the three cases are

Part a $e_D = – 5(140/300) = – 2.3$ $e_S = 4(95/300) = 1.3$

Part d $e_D = – 15(129/258) = – 7.5$ $e_S = 4(84/300) = 1.12$

Part e $e_D = – 5(150/250) = – 3.0$ $e_S = 10(105/250) = 4.20$

Although these elasticity estimates are only approximates, the calculations clearly show that the relative sizes of the elasticities determine the tax burden.

11.6 a. With tax $P_D = P_S + 3$

$1100 – 50P_D = 100P_S – 1000 = 100(P_D – 3) – 1000$

$150P_D = 2400$ $P_D = 16$ $P_S = 13$

$Q = 300$ Total tax = 900

b. Consumers pay 300(16 – 14) = 600

Producers pay 300(14 – 13) = 300

c. PS = .5(300)(13 – 10) = 450, a loss of 350 from Problem 11.2 part d.

Short-run profits = 13(300) – 100C

$C = .5(3)^2 + 30 + 5 = 39.5$

$\pi = 3900 – 3950 = – 50.$

Since total profits were 300, this is a reduction of 350 in short-run profits.

11.7 a. With tax $P_D = P_S + 5.5$

$P_S = 10 + .002Q$

$P_D = 15.5 + .002Q$

$Q = 1050 – 50(15.5 + .002Q) = 275 – .1Q$

$1.1Q = 275$ $Q = 250$ $P_D = 16$ $r = 0.5$

Total tax = 5.5(250) = 1,375

Demanders pay 250(16 – 11) = 1,250

Producers pay 250(11 – 10.5) = 125

b. CS originally = .5(500)(21 – 11) = 2,500

CS now = .5(250)(21 – 16) = 625

PS originally = .5(500)(11 – 10) = 250

PS now = .5(250)(10.5 – 10) = 62.5

c. Loss of rents = .5(250) + .5(250)(.5) = 187.5

This is the total loss of PS in part b. Occurs because the only reason for upward-sloping supply is upward slope of film royalties supply.

11.8 a. Domestic Equilibrium 150P = 5000 – 100P

P = 20 Q = 3000 (i.e., 3 million)

b. Price drops to 10, Q_D = 4000

Domestic production is 150(10) = 1500.

Imports = 2500.

c. Price rises to 15, Q_D = 3500

Domestic production = 150(15) = 2250.

Imports = 1250 Tariff revenues = 6250.

CS with no tariff = .5(4000)(50 – 10) = 80,000

CS with tariff = .5(3500)(50 – 15) = 61,250

Loss = 18,750

Transfer to producers = 5(1500) + .5(2250 – 1500)(15 – 10) = 9,375

Deadweight loss = Total loss – Tariffs – Transfer = 3,125

Check by triangles

Loss = .5(2250 – 1500)(5) + .5(4000 – 3500)(5) = 1875 + 1250 = 3125

d. With quota of 1250, results duplicate part c except no tariff revenues are collected. Now 6250 can be obtained by rent seekers.

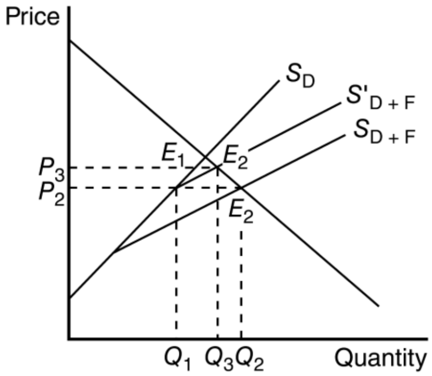
11.9 Price now rises to 9.6.
Q_D = 13.25
Q_S = 12.48. Hence imports fall to 0.77. Total tariff revenues are .462 (billion), a decline from the case in Example 11.3. The deadweight losses increase dramatically, however:

DW_1 = .5(.6)(14.3 – 13.25) = .315

DW_2 = .5(.6)(12.48 – 11.7) = .234

$DW_1 + DW_2$ = .549, an increase of 37 percent from the total loss calculated in Example 11.3.

11.10 a. In the graph D is the demand for importable goods, S_D is the domestic supply curve and S_{D+F} is the supply curve for domestic and foreign goods. Domestic Equilibrium is at E_1, free trade equilibrium is at E_2. Imports are given by $Q_2 - Q_3$.



b. A tariff shifts S_{D+F} to S'_{D+F} = Equilibrium is at E_3. Imports fall and quantity supplied domestically increases.

c., d. Losses of consumer surplus can be illustrated in much the same way as in the infinitely elastic supply case. Gains of domestic producer surplus can also be shown in a way similar to that used previously. In this case, however, some portion of tariff revenue is paid by the foreign producers since the price rise from P_2 to P_3 is less than the amount of the tariff (given by the vertical distance between S'_{D+F} and S_{D+F}). These tariffs may partly affect the deadweight losses of domestic consumer surplus.

CHAPTER 12

GENERAL EQUILIBRIUM AND WELFARE

The problems in this chapter focus primarily on the simple two-good general equilibrium model in which "supply" is represented by the production possibility frontier and "demand" by a set of indifference curves. Because it is probably impossible to develop realistic general equilibrium problems that are tractable, students should be warned about the very simple nature of the approach used here. Specifically, none of the problems does a very good job of tying input and output markets, but it is in that connection that general equilibrium models may be most needed. The Extensions for the chapter provide a very brief introduction to computable general equilibrium models and how they are used.

Problems 12.1–12.5 are primarily concerned with setting up general equilibrium conditions whereas 12.6–12.10 introduce some efficiency ideas. Many of these problems can be best explained with partial equilibrium diagrams of the two markets. It is important for students to see what is being missed when they use only a single-good model.

Comments on Problems

12.1 This problem repeats examples in both Chapter 1 and 12 in which the production possibility frontier is concave (a quarter ellipse). It is a good starting problem because it involves very simple computations.

12.2 A generalization of Example 12.1 that involves computing the production possibility frontier implied by two Cobb-Douglas production functions. Probably no one should try to work out all these cases analytically. Use of computer simulation techniques may offer a better route to a solution (it is easy to program this in Excel, for example). It is important here to see the connection between returns to scale and the shape of the production possibility frontier.

12.3 This is a geometrical proof of the Rybczynski Theorem from international trade theory. Although it requires only facility with the production box diagram, it is a fairly difficult problem. Extra credit might be given for the correct spelling of the discoverer's name.

12.4 This problem introduces a general equilibrium model with a linear production possibility frontier. The price ratio is therefore fixed, but relative demands determine actual production levels. Because the utility functions are Cobb-Douglas, the problem can be most easily worked using a budget-share approach.

12.5 This is an introduction to excess demand functions and Walras' Law.

12.6 This problem uses a quarter-circle production possibility frontier and a Cobb-Douglas utility function to derive an efficient allocation. The problem then proceeds to illustrate the gains from trade. It provides a good illustration of the sources of those gains.

12.7 This is a fixed-proportions example that yields a concave production possibility frontier. This is a good initial problem although students should be warned that calculus-type efficiency conditions do not hold precisely for this type of problem.

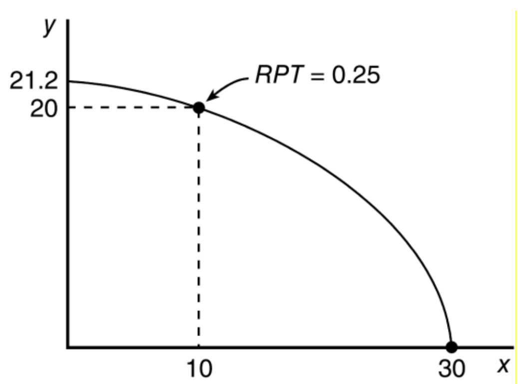
12.8 This provides an example of efficiency in the regional allocation of resources. The problem could provide a good starting introduction to mathematical representations of comparative versus absolute advantage or for a discussion of migration. To make the problem a bit easier, students might be explicitly shown that the production possibility frontier has a particularly simple form for both the regions here (e.g., for region A it is $x^2 + y^2 = 100$).

12.9 This problem provides a numerical example of an Edgeworth Box in which efficient allocations are easy to compute because one individual wishes to consume the goods in fixed proportions.

12.10 A continuation of Problem 12.9 that illustrates notions of the core and exchange offer curves.

Solutions

12.1 a.



b. $9x^2 = 900$; $x = 10$, $y = 20$

c. If $x = 9$ on the production possibility frontier,

$$y = \sqrt{\frac{819}{2}} = 20.24$$

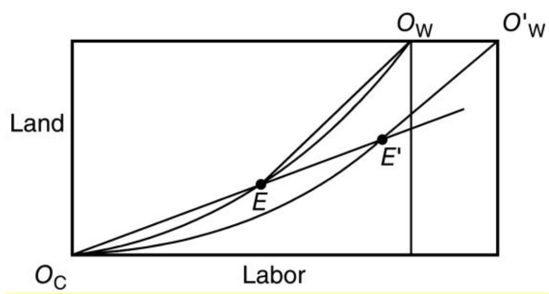
If $x = 11$ on the frontier, $y = \sqrt{\frac{779}{2}} = 19.74$.

Hence, *RPT* is approximately $-\frac{\Delta y}{\Delta x}=\frac{-(-0.50)}{2}=.25$.

12.2 I have never succeeded in deriving an analytical expression for all these cases. I have, however, used computer simulations (for example with Excel) to derive approximations to these production possibility frontiers. These tend to show that increasing returns to scale is compatible with concavity providing factor intensities are suitably different (case [e]), but convexity arises when factor intensities are similar (case [d]).

12.3 a. Draw the production possibility frontier and the Edgeworth box diagram. Find where *P* line is tangent to *PPF*; then go back to the box diagram to find input ratio. See Corn Law Debate example in the text.

b. *P* given, land/labor ratio is constant.



Equilibrium moves from *E* to *E* '.

Cloth $(O_C E' > O_C E)$ Wheat $(O_W' E' < O_W E)$

12.4 a. $p_x / p_y = 3/2$

b. If wage = 1, each person’s income is 10. Smith spends 3 on *x*, 7 on *y*.

Jones spends 5 on *x*, 5 on *y*.

Since $\frac{x}{2}+\frac{y}{3}=20$, and demands are $x=\frac{8}{p_x}$, $y=\frac{12}{p_y}$

we have $\frac{8}{2p_x}+\frac{12}{3p_y}=\frac{8}{2p_x}+\frac{12}{2p_x}=20$, or $p_x=1/2, p_y=1/3$

So Smith demands 6*x*, 21*y*.

Jones demands 10*x*, 15*y*.

c. Production is *x* = 16, *y* = 36.

20 hours of labor are allocated: 8 to *x* production, 12 to *y* production.

12.5 a. Functions are obviously homogeneous of degree zero since doubling of p_1, p_2 and p_3 does not change ED_2 or ED_3.

b. Walras’s Law states $\sum_i p_i \; ED_i = 0$

Hence, if $ED_2 = ED_3 = 0$, $p_1 ED_1 = 0$ or $ED_1 = 0$.

Can calculate ED_1 as $p_1ED_1 = -p_2ED_2 - p_3ED_3$

$$ED_1 = [3p_2^2 - 6p_2p_3 + 2p_3^2 + p_1p_2 + 2p_1p_3]/p_1^2 .$$

Notice that ED_1 is homogeneous of degree zero also.

c. $ED_2 = 0$ and $ED_3 = 0$ can be solved simultaneously for p_2/p_1 and p_3/p_1.

Simple algebra yields $p_2/p_1 = 3$ $\quad p_3/p_1 = 5$.

If set $p_1 = 1$ have $p_2 = 3$, $p_3 = 5$ at these absolute prices

$ED_1 = ED_2 = ED_3 = 0$.

12.6 $PPF = f^2 + c^2 = 200$

$$RPT = -\frac{dc}{df} = \frac{f}{c} \quad MRS = \frac{\partial U/\partial f}{\partial U/\partial c} = \frac{0.5U/f}{0.5U/c} = \frac{c}{f}$$

a. For efficiency, set $MRS = RPT$ $\quad f/c = c/f$ or $f = c$

PPF: $2c^2 = 200$, $c = 10 = f = U$, $RPT = 1$.

b. Demand: $P_F/P_C = 2/1 = MRS = c/f$ so $c = 2f$.

Budget: $2f + 1c = 30$ the value of production. Substituting from the demand equation: $4f = 30$ $\quad f = 30/4$, $c = 15$.

$U = \sqrt{15 \cdot 30/4} = \sqrt{112.5}$; an improvement from (a) (the "demand effect").

c. Set $RPT = 2/1$ $\quad f = 2c$.

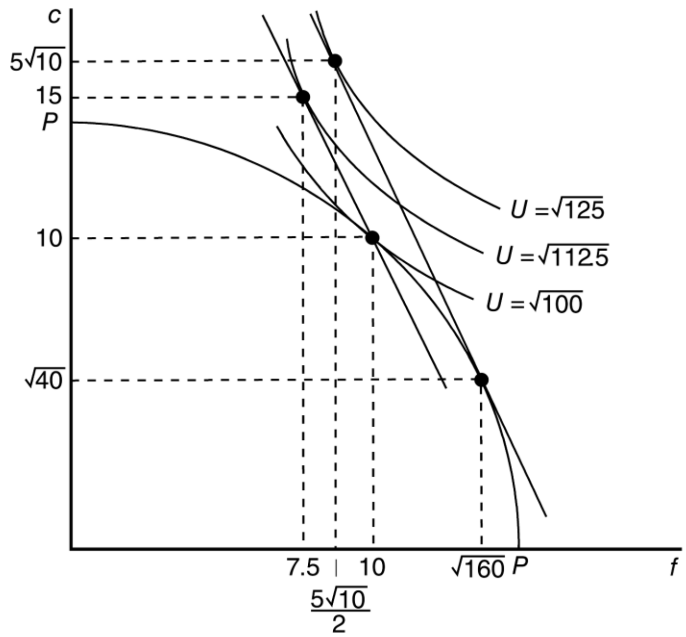
PPF: $5c^2 = 200$, $c = \sqrt{40}$, $f = \sqrt{160}$

Budget now is: $2\sqrt{160} + 1\sqrt{40} = 5\sqrt{40} = 10\sqrt{10}$

Spend $5\sqrt{10}$ on f and $5\sqrt{10}$ on c.

$$c = 5\sqrt{10}, \quad f = \frac{5\sqrt{10}}{2} \quad U = \sqrt{125}:$$

A further improvement ("the production specialization effect")

d. 

12.7 f = Food c = Cloth

a. Labor constraint $f + c = 100$ (see graph below)

b. Land constraint $2f + c = 150$ (see graph below)

c. Heavy line in graph below satisfies both constraints.

d. Concave because it must satisfy both constraints. Since the *RPT* = 1 for the labor constraint and 2 for the land constraint, the production possibility frontier of part (c) exhibits an increasing *RPT*; hence it is concave.

e. Constraints intersect at $f = 50$, $c = 50$.

$$f < 50 \quad \frac{dc}{df} = -1 \quad \text{so} \quad \frac{p_f}{p_c} = 1$$

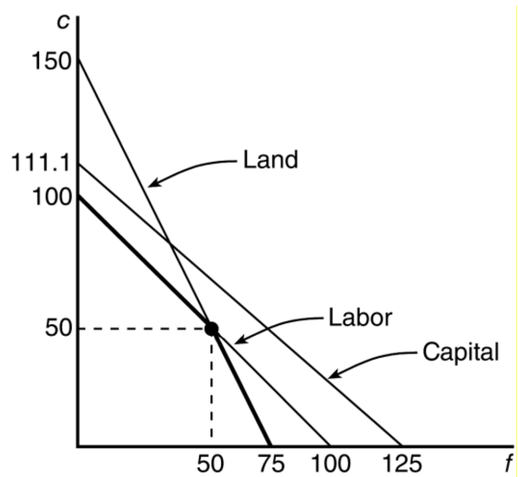
$$f > 50 \quad \frac{dc}{df} = -2 \quad \text{so} \quad \frac{p_f}{p_c} = 2$$

f. If for consumers $\frac{dc}{df} = -\frac{5}{4}$ so $\frac{p_f}{p_c} = \frac{5}{4}$.

g. If $p_f/p_c = 1.9$ or $p_f/p_c = 1.1$, will still choose $f = 50$, $c = 50$ since both price lines "tangent" to production possibility frontier at its kink.

h. $.8f + .9c = 100$

Capital constraint: $c = 0$ $f = 125$ $f = 0$ $c = 111.1$



Same PPF since capital constraint is nowhere binding.

12.8 a. $x_A^2 = L_{x_A},\ y_A^2 = L_{y_A},\quad x_A^2 + y_A^2 = L_{x_A} + L_{y_A} = 100$

Same for region B so $4x_B^2 + 4y_B^2 = 100$

b. *RPT*'s should be equal.

c. $RPT_A = -\dfrac{dy}{dx} = \dfrac{x_A}{y_A} \qquad RPT_B = -\dfrac{dy}{dx} = \dfrac{x_B}{y_B}$

Therefore, $\dfrac{x_A}{y_A} = \dfrac{x_B}{y_B}$, hence $y_A^2 = x_A^2\left(\dfrac{y_B^2}{x_B^2}\right)$.

But $x_A^2 + y_A^2 = 4(x_B^2 + y_B^2)$ so substituting for Y_A^2 yields

$$x_A^2\left(1+\frac{y_B^2}{x_B^2}\right) = 4x_B^2\left(1+\frac{y_B^2}{x_B^2}\right)$$

$x_A = 2x_B$ also $y_A = 2y_B$

$x_T = x_A + x_B = 3x_B \qquad x_T^2 = 9x_B^2$

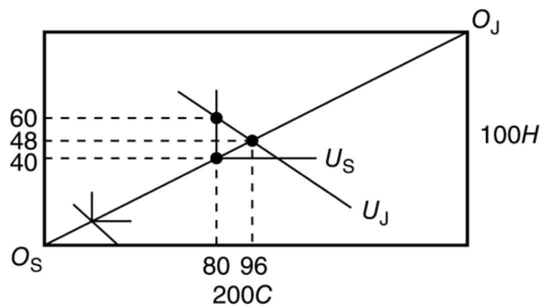
$y_T = y_A + y_B = 3y_B \qquad y_T^2 = 9y_B^2$

$x_T^2 + y_T^2 = 9(x_B^2 + y_B^2) = 9\cdot 100/4 = 225$

If $x_T = 12 \qquad x_T^2 = 144 \qquad y_T = \sqrt{225-14} = 9$

Note: Can also show that more of both goods can be produced if labor could move between regions.

12.9



a. Contract curve is straight line with slope of 2. The only price ratio in equilibrium is 3 to 4 (p_c to p_h).

b. 40h, 80c is on C.C. Jones will have 60h and 120c.

c. 60h, 80c is not on *C.C*. Equilibrium will be between 40h, 80c (for Smith) and 48h, 96c (for Smith), as Jones will not accept any trades that make him worse off. $U_J = 4(40) + 3(120) = 520$. This intersects the contract curve at $520 = 4(h) + 3(2h)$, $h = 52$, $c = 104$.

d. Smith grabs everything; trading ends up at O_J on *C.C*.

12.10 (for diagram, see Problem 12.9)

a. Core is O_S, O_J between points *A* and *B*.

b. Offer curve for Smith is portion of $O_S\ O_J$ above point *A* (since requires fixed proportions). For Jones, offer curve is to consume only *C* for $p_c/p_h < 3/4$ and consume only *h* for $p_c/p_h > 3/4$. For $p_c/p_h = 3/4$, offer curve is the indifference curve U_J.

c. Only equilibrium is at point *B*. $p_c/p_h = 3/4$ and Smith gets all the gains from trade--the benefits of being inflexible.

CHAPTER 13

MONOPOLY

The problems in this chapter deal primarily with marginal revenue-marginal cost calculations in different contexts. For such problems, students' primary difficulty is to remember that the marginal revenue concept requires differentiation with respect to *quantity*. Often students choose to differentiate total revenue with respect to price and then get very confused on how to set this equal to marginal cost. Of course, it is possible to phrase the monopolist's problem as one of choosing a profit-maximizing price, but then the inverse demand function must be used to derive a marginal cost expression. The other principal focus of some of the problems in this chapter is consumer's surplus. Because the computations usually involve linear demand curves, they are quite straightforward.

Comments on Problems

13.1 A simple marginal revenue–marginal cost and consumer surplus computation.

13.2 An example of the $MR = MC$ calculation with three different types of cost curves.

13.3 An example of the $MR = MC$ calculation with three different demand and marginal revenue curves. Illustrates the "inverse elasticity" rule.

13.4 Examines graphically the various possible ways in which shift in demand may affect the market equilibrium in a monopoly.

13.5 Introduces advertising expenditures as a choice variable for a monopoly. The problem also asks the student to view market price as the decision variable for the monopoly.

13.6 This problem examines taxation of monopoly output. It shows that some results from competitive tax-incidence theory do not carry over.

13.7 A price discrimination example in which markets are separated by transport costs. The problem shows how the price differential is constrained by the extent of those costs. Part d asks students to consider a simple two-part tariff.

13.8 A marginal revenue-marginal cost computation for the case in which monopolist's costs exceed those of a perfect competitor. The problem suggests that the social losses from such increased costs may be of the same order of magnitude as the deadweight loss from monopolization.

13.9 This problem examines some issues in the design of subsidies for a monopoly.

13.10 A problem involving quality choice. Shows that in this case, monopolist's and competitive choices are the same (though output levels differ).

Solutions

13.1 a. $P = 53 - Q \qquad PQ = 53Q - Q^2$

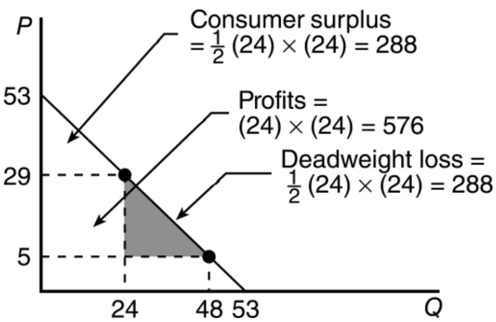
$MR = 53 - 2Q = MC = 5$

$Q = 24, \quad P = 29, \quad \pi = (P - AC) \cdot Q = 576$

b. $MC = P = 5 \qquad P = 5, \; Q = 48$

c. Competitive Consumers' Surplus = $2(48)^2 = 1152$.

Under monopoly:



Notice that the sum of consumer surplus, profits, and deadweight loss under monopoly equals competitive consumer surplus.

13.2 Market demand $Q = 70 - P$, $MR = 70 - 2Q$.

a. $AC = MC = 6$. To maximize profits set $MC = MR$.

$6 = 70 - 2Q \qquad Q = 32 \; P = 38$

$\pi = (P - AC) \cdot \; Q = (38 - 6) \;\; 32 = 1024$

b. $C = .25Q^2 - 5Q + 300$, $MC = .5Q - 5$. Set $MC = MR$
$.5Q - 5 = 70 - 2Q \qquad Q = 30 \qquad P = 40$

$\pi = TR - TC = (30)(40) - [.25(30)^2 - 5\,(30) + 300] = 825.$

c. $C = .0133Q^3 - 5Q + 250$. $MC = .04Q^2 - 5$

$MC = MR$

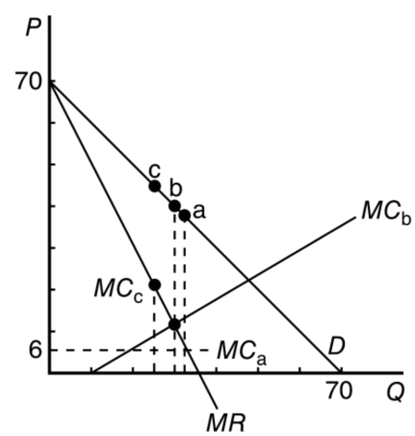
Therefore: $04Q^2 - 5 = 70 - 2Q$ or
$.04Q^2 + 2Q - 75 = 0$.

Quadratic formula gives $Q = 25$.

If $Q = 25$, $P = 45$

$R = 1125$

$C = 332.8 \quad (MC = 20) \quad \pi = 792.2$



13.3 a. $AC = MC = 10$, $Q = 60 - P$, $MR = 60 - 2Q$.

For profit maximum, $MC = MR$ $10 = 60\text{-}2Q$ $Q = 25$ $P = 35$

$\pi = TR - TC = (25)(35) - (25)(10) = 625$.

b. $AC = MC = 10$, $Q = 45 - 5P$, $MR = 90 - 4Q$.

For profit maximum, $MC = MR$ $10 = 90 - 4Q$ $Q = 20$ $P = 50$

$\pi = (20)(50) - (20)(10) = 800$.

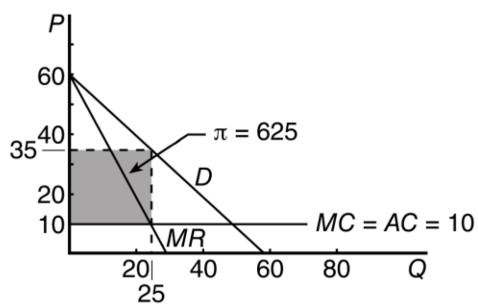
c. $AC = MC = 10$, $Q = 100 - 2P$, $MR = 50 - Q$.

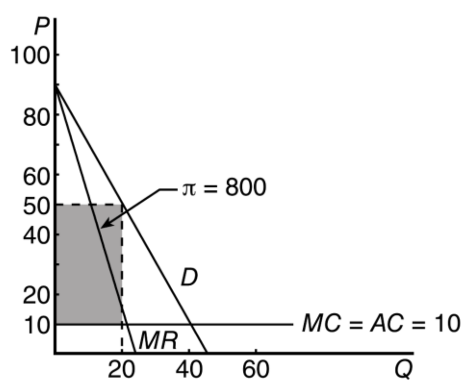
For profit maximum, $MC = MR$ $10 = 50 - Q$ $Q = 40$ $P = 30$.

$\pi = (40)(30) - (40)(10) = 800$.

Note: Here the inverse elasticity rule is clearly illustrated:

Problem Part	$e = \frac{\partial Q}{\partial P} \cdot \frac{P}{Q}$	$\frac{-1}{e_{Q,P}} = \frac{P - MC}{P}$
a	$-1(35/25) = -1.4$	$.71 = (35 - 10)/35$
b	$-.5(50/20) = -1.25$	$.80 = (50 - 10)/50$
c	$-2(30/40) = -1.5$	$.67 = (30 - 10)/30$

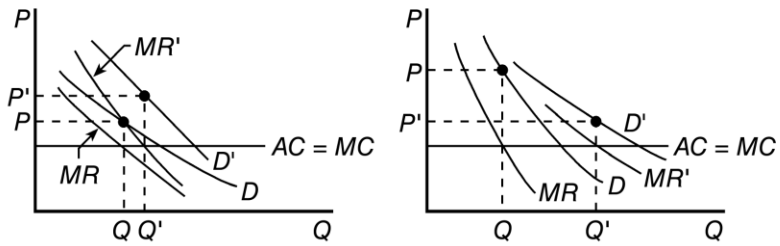




d.

The supply curve for a monopoly is a single point, namely, that quantity-price combination which corresponds to the quantity for which $MC = MR$. Any attempt to connect equilibrium points (price-quantity points) on the market demand curves has little meaning and brings about a strange shape. One reason for this is that as the demand curve shifts, its elasticity (and its *MR* curve) usually changes bringing about widely varying price and quantity changes.

13.4 a.



(case 1) (case 2)

b. There is no supply curve for monopoly; have to examine *MR* = *MC* intersection because any shift in demand is accompanied by a shift in *MR* curve. Case (1) and case (2) above show that *P* may rise or fall in response to an increase in demand.

c. Can use inverse elasticity rule to examine this

$$-e = \frac{P}{P - MC} = \frac{P}{P - MR}.$$

As $-e$ falls toward 1 (becomes less elastic), $P - MR$ increases.

<u>Case 1</u> *MC* constant so profit-maximizing *MR* is constant

If $-e\downarrow,\quad P - MR\uparrow\quad P\uparrow.$

If $-e$ constant, $P - MR$ constant, P constant

If $-e\uparrow, P - MR\downarrow, P\downarrow.$

<u>Case 2</u> *MC* falling so profit-maximizing *MR* falls:

If $-e\downarrow,\quad P - MR\uparrow\quad P$ may rise or fall

If $-e$ constant, $P - MR$ constant, $P\downarrow$

If $-e\uparrow, P - MR\downarrow, P\downarrow$

<u>Case 3</u> *MC* rising so profit-maximizing *MR* must increase

If $-e\downarrow,\quad P - MR\uparrow\quad P\uparrow$

If $-e$ constant, $P - MR$ constant, $MR\uparrow, P\uparrow$

If $-e\uparrow,\quad P - MR\downarrow\quad P$ may rise or fall

13.5 $Q = (20 - P)(1 + .1A - .01A^2)$

Let $K = 1 + .1A + .01A^2$ $\quad \dfrac{dK}{dA} = .1 - .02A$

$\pi = PQ - C = (20P - P^2)K - (200 - 10P)K - 15 - A$

$\dfrac{\partial \pi}{\partial P} = (20 - 2P)K + 10K = 0.$

a. $20 - 2P = -10$ $\quad P = 15$ regardless of K or A

If $A = 0$, $Q = 5$, $C = 65$ $\quad \pi = 10$

b. If $P = 15$, $\quad \pi = 75K - 50K - 15 - A = 25K - 15 - A = 10 + 1.5A - 0.25A^2$

$\dfrac{\partial \pi}{\partial A} = 1.5 - 0.5A = 0 \quad so \quad A = 3$

$Q = 5(1 + .3 - .09) = 6.05$

$PQ = 90.75$ $\quad$ C = 60.5 + 15 + 3 = 78.5

$\pi = 12.25$; this represents an increase over the case $A = 0$.

13.6 The inverse elasticity rule is $P = MC/(1 + 1/e)$. When the monopoly is subject to an *ad valorem* tax of t, this becomes $P = \dfrac{MC}{(1-t)} \cdot \dfrac{1}{1 + \frac{1}{e}}$.

a. With linear demand, e falls (becomes more elastic) as price rises. Hence,

$$P_{aftertax} = \frac{MC}{(1-t)} \cdot \frac{1}{1 + \frac{1}{e_{aftertax}}} < \frac{MC}{(1-t)} \cdot \frac{1}{1 + \frac{1}{e_{pretax}}} = \frac{P_{pretax}}{(1-t)}$$

b. With constant elasticity demand, the inequality in part a becomes an equality so $P_{after\,tax} = P_{pretax}/(1-t)$.

c. If the monopoly operates on a negatively sloped portion of its marginal cost curve we have (in the constant elasticity case)

$$P_{aftertax} = \frac{MC_{aftertax}}{(1-t)} \cdot \frac{1}{1 + \frac{1}{e}} > \frac{MC_{pretax}}{(1-t)} \cdot \frac{1}{1 + \frac{1}{e}} = \frac{P_{pretax}}{(1-t)} .$$

d. The key part of this question is the requirement of equal tax revenues. That is, $tP_aQ_a = \tau Q_s$ where the subscripts refer to the monopoly's choices under the two tax regimes. Assuming constant MC, profit maximization requires

$MC = P_a(1-t) \cdot \dfrac{1}{1 + \frac{1}{e}} = P_s \cdot \dfrac{1}{1 + \frac{1}{e}} - \tau$. Combining this with the revenue neutrality condition shows $P_s > P_a$.

13.7 a. $Q_1 = 55 - P_1$ $\quad R_1 = (55 - Q_1)Q_1 = 55Q_1 - Q_1^2$

$MR_1 = 55 - 2Q_1 = 5 \quad Q_1 = 25, P_1 = 30$

$Q_2 = 70 - 2P_2 \quad R_2 = \left(\frac{70 - Q_2}{2}\right) \cdot Q_2 = (70Q_2 - Q_2^2)/2$

$MR = 35 - Q_2 = 5 \quad Q_2 = 30, P_2 = 20$

$\pi = (30 - 5)25 + (20 - 5)30 = 1075$

b. Producer wants to maximize price differential in order to maximize profits but maximum price differential is \$5. So $P_1 = P_2 + 5$.

$\pi = (P_1 - 5)(55 - P_1) + (P_2 - 5)(70 - 2P_2)$

Set up Lagrangian $? = \pi + \lambda\ 5 - P_1 + P_2)$

$$\frac{\partial £}{\partial P_1} = 60 - 2P_1 - \lambda = 0$$

$$\frac{\partial £}{\partial P_2} = 80 - 4P_2 + \lambda = 0$$

$$\frac{\partial £}{\partial \lambda} = 5 - P_1 + P_2 = 0$$

Hence $60 - 2P_1 = 4P_2 - 80$ and $P_1 = P_2 + 5$.

$130 = 6P_2 \quad P_2 = 21.66 \quad P_1 = 26.66 \quad \pi = 1058.33$

c. $P_1 = P_2$ So $\pi = 140P - 3P^2 - 625 \qquad \frac{\partial \pi}{\partial P} = 140 - 6P = 0$

$P = \frac{140}{6} = 23.33 \quad Q_1 = 31.67 \quad Q_2 = 23.33 \quad \pi = 1008.33$

d. If the firm adopts a linear tariff of the form $T(Q_i) = \alpha_i + mQ_i$, it can maximize profit by setting $m = 5$,

$\alpha_1 = .5(55 - 5)(50) = 1250$

$\alpha_2 = .5(35 - 5)(60) = 900$

and $\pi = 2150$.

Notice that in this problem neither market can be uniquely identified as the "least willing" buyer so a solution similar to Example 13.5 is not possible. If the entry fee were constrained to be equal in the two markets, the firm could set $m = 0$, and charge a fee of 1225 (the most buyers in market 2 would pay). This would yield profits of 2450 – 125(5) = 1825 which is inferior to profits yielded with $T(Q_i)$.

13.8 a. For perfect competition, $MC = \$10$. For monopoly $MC = \$12$.

$Q_D = 1000 - 50P$. The competitive solution is $P = MC = \$10$. Thus $Q = 500$.

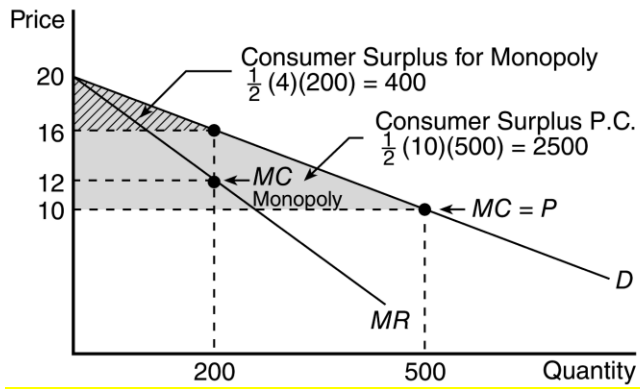
Monopoly: $P = 20 - \frac{1}{50}Q$, $PQ = 20Q - \frac{1}{50}Q^2$

Produce where $MR = MC$. $MR = 20 - \frac{1}{25}Q = 12$. $Q = 200$, $P = \$16$

b. See graph below.

Loss of consumer surplus = Competitive CS – monopoly CS =

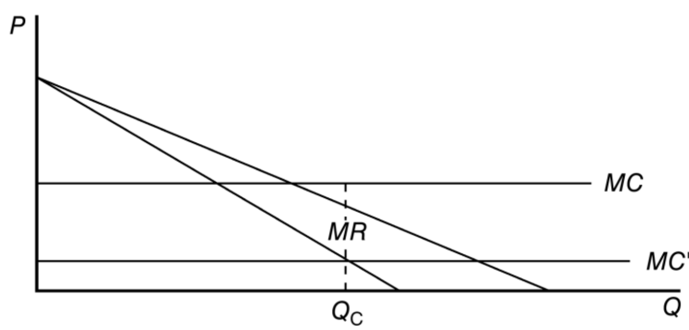
2500 – 400 = 2100.

c. 

Of this 2100 loss, 800 is a transfer into monopoly profit, 400 is a loss from increased costs under monopoly, and 900 is a "pure" deadweight loss.

13.9 a. The government wishes the monopoly to expand output toward $P = MC$. A lump-sum subsidy will have no effect on the monopolist's profit maximizing choice, so this will not achieve the goal.

b. A subsidy per unit of output will effectively shift the MC curve downward. The figure illustrates this for the constant MC case.



c. A subsidy (t) must be chosen so that the monopoly chooses the socially optimal quantity, given t. Since social optimality requires $P = MC$ and profit maximization requires that $MR = MC - t =$

$$P\left(1+\frac{1}{e}\right),$$

substitution yields $\frac{t}{P} = -\frac{1}{e}$ as was to be shown.

Intuitively, the monopoly creates a gap between price and marginal cost and the optimal subsidy is chosen to equal that gap expressed as a ratio to price.

13.10 Since consumers only value $X \cdot Q$, firms can be treated as selling that commodity (i.e., batteries of a specific useful life). Firms seek to minimize the cost of producing $X \cdot Q$ for any level of that output. Setting up the Lagrangian, $? = C\ \ X)Q + \lambda(K - XQ)$ yields the following first order conditions for a minimum:

$$\frac{\partial £}{\partial X} = C'(X)Q - \lambda Q = 0$$

$$\frac{\partial £}{\partial Q} = C(X) - \lambda X = 0$$

$$\frac{\partial £}{\partial \lambda} = K - XQ = 0$$

Combining the first two shows that $C(X) - C'(X)X = 0$ or $X = \frac{C\ \ (X)}{C'(X)}$.

Hence, the level of X chosen is independent of Q (and of market structure). The nature of the demand and cost functions here allow the durability decision to be separated from the output-pricing decision. (This may be the most general case for which such a result holds.)

CHAPTER 14

TRADITIONAL MODELS OF IMPERFECT COMPETITION

The problems in this chapter are of two types: analytical and essay. The analytical problems look at a few special cases of imperfectly competitive markets for which tractable results can be derived. Some of these results (especially those in Problems 14.4, 14.5, and 14.6) are quite important in the industrial organization literature. The essay problems in the chapter (14.3 and 14.8) do not offer such definitive results but instead ask students to think a bit more broadly about some institutional issues in industrial organizations.

Comments on Problems

14.1 This is a simple duopoly problem that duplicates Example 14.1 with different numbers.

14.2 A problem providing numerical solutions for monopoly and Cournot equilibria for the simple linear demand curve and constant marginal cost case. The problem shows that in this case the competitive solution ($P = 5$) is the limit of the Cournot outcomes as the number of firms approaches infinity.

14.3 An essay-type question that seeks to explore some purported empirical observations in various markets.

14.4 A problem that shows the derivation of the "Dorfman-Steiner" conditions for optimal spending on advertising.

14.5 The problem shows that the widely-used Herfindahl Index is correlated with industry profitability, if the firms in industry follow Cournot pricing strategies.

14.6 A problem based on Salop's "circular" model of demand. This is a very useful model both for spatial applications and for looking at issues in product differentiation.

14.7 This problem provides a numerical example of price leadership. Construction of the net demand curve provides a good illustration of the assumptions behind the behavior of the "competitive fringe."

14.8 An essay question about monopoly and innovation. The question is a very complex one in reality though the solutions provide Fellner's suggested simple answer to the problem. This might be contrasted to Schumpeter's views which are summarized at the end of Chapter 13.

14.9 An example of contestability in the natural monopoly context. Computations here do not work evenly—for an approximation, see the solutions below.

Solutions

14.1 $Q = 150 - P$ $MC = 0$

a. A zero cost monopolist would produce that output for which MR is equal to 0 ($MR = MC = 0$). $MR = 0$ at one half of the demand curve's horizontal intercept. Therefore, $Q = 75$ $P = 75$ $\pi = 5625$

b. $q_1 + q_2 = 150 - P$

Demand curve for firm 1: $q_1 = (150 - q_2) - P$

Profit maximizing output level: $q_1 = \frac{150 - q_2}{2}$

Demand curve for firm 2: $q_2 = (150 - q_1) - P$

Profit maximizing output level: $q_2 = \frac{150 - q_1}{2}$

Market equilibrium: $q_1 = [150 - (150 - q_1)/2] - 2$

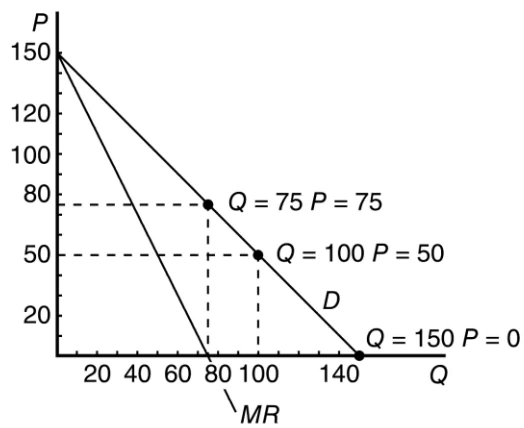
$$q_1 = \frac{150 - 75 + q_1/2}{2} = 37.5 + \frac{q_1}{4}$$

$4q_1 = 150 + q_1$ $3q_1 = 150$ $q_1 = 50$, $q_2 = 50$, $P = 50$

$\pi_1 = \pi_2 = \$2{,}500$. $\pi_{total} = 5{,}000$.

c. Under perfect competition, $P = MC = 0$.

$Q = 150$, $P = 0$, $\pi = 0$.



14.2 a. $Q = 53 - P$ $P = 53 - Q$ $PQ = 53Q - Q^2$

$MR = 53 - 2Q = MC = 5$ $Q = 24$, $P = 29$, $\pi = 576$

b and c. $P = 53 - q_1 - q_2$ $Pq_1 = 53q_1 - q_1^2 - q_1q_2$

$\pi = Pq_1 - 5q_1 = 48q_1 - q_1^2 - q_1q_2$

$$\frac{\partial \pi}{\partial q_1} = 48 - 2q_1 - q_2 = 0$$

$q_1 = (24 - q_2)/2$ Similarly, $q_2 = (24 - q_1)/2$

d. $q_1 = (24 - q_2)/2 = (48 - 24 + q_1)/4 = 6 + \frac{q_1}{4}$

$3/4\, q_1 = 6$ $q_1 = 8 = q_2$

e. $P = 37$ $\pi_1 = \pi_2 = 256$ Total $\pi = 512$

f. $P = 53 - q_1 - q_2 \ldots - q_n$

$$\pi_1 = 53q_1 - q_1^2 - q_1 \sum_{i=2}^{n} q_i - 5q_1$$

Argue by symmetry: $q_1 = q_2 = \ldots = q_n$.

$$\pi_1 = 53q_1 - q_1^2 - q_1(n-1)q_i - 5q_1$$

$$\frac{\partial \pi_1}{\partial q_1} = 48 - 2q_1 - (n-1)q_i = 0$$

$$q_1 = 24 - \frac{(n-1)q_i}{2} \quad \text{Let } q_i = q_1 \quad q_1 = \frac{48}{n+1}$$

g. $P = 53 - \frac{n}{n+1}48$ As $n \to \infty, P \to 5, \pi \to 0.$

The model yields competitive results as n gets large.

14.3 a. Price leadership.

b. Product differentiation strategies. This facilitates price discrimination (assuming one model does not ultimately triumph because of network externalities).

c. Maintain market share; perhaps act as loss leader to help them sell other types of policies. May also just be improper accounting (insurance companies have a way of forgetting about the returns they make on their investments when making such statements).

d. Competition from the Japanese in the most likely answer—though the improvement in U.S. quality may be more fiction than fact.

14.4 Total profits are given by $\pi = pq(P,z) - g(q) - z$.

First-order conditions for a maximum are

$$\frac{\partial \pi}{\partial p} = p\frac{\partial q}{\partial p} + q - g'\frac{\partial q}{\partial p} = 0$$

$$\frac{\partial \pi}{\partial z} = p\frac{\partial q}{\partial z} - g'\frac{\partial q}{\partial z} - 1 = 0$$

Hence $p - g' = \dfrac{\frac{-q}{\partial q}}{\partial p}$ and $p - g' = \dfrac{1}{\frac{\partial q}{\partial z}}$

Or $\dfrac{1}{q} = -\dfrac{\partial q/\partial z}{\partial q/\partial p}$.

Multiplication of this expression by *z/p* gives the required result.

14.5 Equation 14.10 shows that under Cournot competition

$$p + q_i\frac{\partial P}{\partial q_i} \cdot MC = 0 .$$

With constant returns to scale, profits for the i^{th} firm are given by

$$\pi_i = (p - MC\ \ )q_i = -q_i^2\frac{\partial p}{\partial q_i}$$

Dividing by total industry revenue (*pq*) yields

$\dfrac{\pi_i}{pq} = -q_i^2\dfrac{\partial p}{\partial q_i} \cdot \dfrac{1}{pq}$. Multiplying by $\dfrac{p^2 q}{p^2 q}$ yields

$= -\dfrac{(pq_i)^2}{(pq)^2}\dfrac{\partial p}{\partial q_i} \cdot \dfrac{q}{p} = \alpha_i^2\ / e_{q,p}$. (Note, here $\dfrac{\partial p}{\partial q_i} = \dfrac{\partial p}{\partial q}$)

Summing over *i* gives $\dfrac{\sum \pi_i}{pq} = \dfrac{H}{|e_{q,p}|}$

14.6 a. Because the circumference is 1.0 in length, firms are located at intervals of 1/*n*. What any one firm can charge (*p*) is constrained by what its nearest neighbor charges (*p**). Let *x* represent the distance a buyer must travel ($0 \le x \le 1/n$). Travel cost to the first firm is *tx*, to its neighbor is $t(1/n - x)$. Hence, the equation in the text must hold.

b. Because $x = \dfrac{p^* - p}{2t} + \dfrac{1}{2n}$, any one firm's sales are $2x = \dfrac{p^* - p}{t} + \dfrac{1}{n}$, total profits are given by

$$\pi = (p - c)\left[\frac{p^* - p}{t} + \frac{1}{n}\right] - f$$

and the first-order condition for a maximum is:

$$\frac{\partial \pi}{\partial p} = (p-c)(-1/t) + \left[\frac{p^*-p}{t} + \frac{1}{n}\right] = 0 \quad or \quad p = (p^* + c + t/n)/2$$

c. Using the symmetry condition $p = p^*$ yields $p = c + t/n$. Intuitively, each firm charges marginal cost plus a distance-related charge. Any one firm could encroach on the other's consumers by charging less than this, but the extra revenue gained would fall short of c.

d. Because each firm sells $q = 1/n$, $\pi = pq - TC = \frac{c}{n} + \frac{t}{n^2} - \frac{c}{n} - f = \frac{t}{n^2} - f$.

e. Free entry yields a zero-profit equilibrium—hence $n = \sqrt{\frac{t}{f}}$.

f. Because marginal costs are a total of c, no matter how many firms there are, a social optimum would seek to minimize fixed costs plus distance costs:

Total costs $= nf + 2n\int_0^{1/2n} tx\,dx = nf + \frac{t}{4n}$ and these are minimized when $n = \sqrt{\frac{t}{4f}}$

which is indeed half the number calculated in part e.

14.7 a. $Q_D = -2000P + 70{,}000$

1000 firms $MC = q + 5$

Price Taker: set $MC = P$, $q = P - 5$

Total $Q_S = \sum_1^{1000} q = 1000P - 5000$

At equilibrium, $Q_D = Q_S$:

$-2000P + 70{,}000 = 1000P - 5000$

$3000P = 75{,}000$, $P = \$25$, $Q = 20{,}000$.

b. Leader has $MC = AC = 15$.

Demand for Leader $= -2000P + 70{,}000 - Q_S$ (fringe)

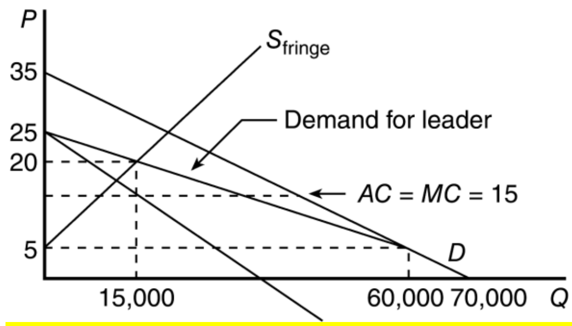
$= -2000P + 70{,}000 - (1000P - 5000)$

$= -3000P + 75{,}000$.

Hence $P = \frac{-Q}{3000} + 25$, $PQ = \frac{-Q^2}{3000} + 25Q$

$MR = \frac{-Q}{1500} + 25 = MC = 15$.

Therefore, Q for Leader = 15,000 $P = 20$ Total $Q_D = 30{,}000$.

c. 

Consumer Surplus

For $P = 25$ $c.s. = 100{,}000$

For $P = 20$ $c.s. = 225{,}000$

For $P = 15$ $c.s. = 400{,}000$

14.8 Fellner gives the following analysis:

$MR_C = P_C$ is demand facing competitive firm.

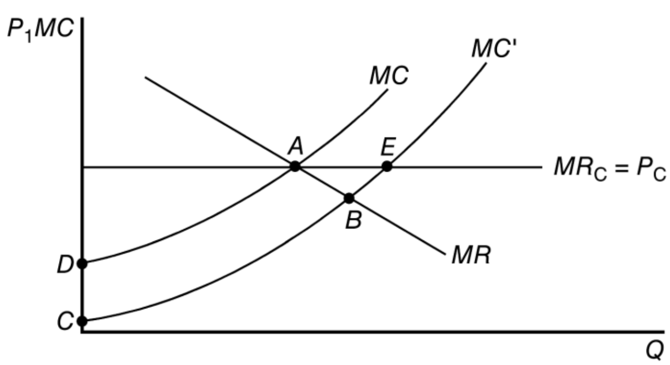
MR is the marginal revenue curve for the monopolist.

Innovation shifts MC to MC'.

Potential profits for competitive firms are $CDAE$.

For monopoly $CDAB$.

Hence, innovation is more profitable for the competitive firm. It is more likely to adopt the innovation.



This analysis neglects a variety of strategic issues about adoption and financing of new technology, however.

14.9 a. Since $MC = \dfrac{\partial C}{\partial Q} = \dfrac{50}{0.1Q - 20}$ is diminishing for $Q > 200$, this industry does exhibit decreasing average and marginal costs and therefore is a natural monopoly.

b. Since $Q = 1000 - 50P$,

$P = 20 - Q/50$ $PQ = 20Q - Q^2/50$

$MR = 20 - Q/25.$

Profit maximization requires

$MR = 20 - Q/25 = MC = 50/(.1Q - 20)$

$-.004Q^2 + 2.8Q - 400 = 50$

$Q^2 - 700Q + 112{,}500 = 0.$

Applying the binomial formula yields $Q = \frac{700 \pm 200}{2}$ so $Q = 450$ is the profit maximizing output.

At $Q = 450$, $P = 11$

$R = 4{,}950$ $C = 500 \ln 25 = 1{,}609$ $\pi = 3{,}341.$

c. To deter entry need $P = AC$

$20 - Q/50 = \frac{500 \ln(.1Q - 20)}{Q}$

I have only been able to attain an approximate solution here. If $Q = 880$, $P = 2.4$, $AC = 500 \ln(68)/880 = 2.397.$

This is the approximate contestable solution. Notice how far it is from the monopoly outcome.

CHAPTER 15

GAME THEORY MODELS OF PRICING

The first six problems for this chapter are intended to illustrate the concept of Nash equilibrium in a variety of contexts. Many of them have only modest economic content, but are traditional game theory problems. The remaining problems (15.7–15.12) in the chapter show how game theory tools can be applied to models of pricing. Many of these represent extensions or generalizations of the results illustrated in Chapter 14.

Comments on Problems

15.1 The classic "Stag Hunt" game attributed to Rousseau. The most interesting aspect of the game is the decline in the value of cooperation as the number of players expands.

15.2 A simple game with continuous strategies in which there are multiple Nash equilibria.

15.3 A continuation of Example 15.2 that shows how mixed strategy equilibria depend on the payoffs to "The Battle of the Sexes" game.

15.4 This is a problem based on Becker's famous "Rotten Kid Theorem." The problem provides a good illustration of backward induction.

15.5 The "Chicken" game. This game illustrates the importance of credible threats and pre-commitments.

15.6 An illustration of an auction game. A more detailed example from auction theory is provided in problem 15.12.

15.7 An illustration of how competitive results do not arise in Bertrand games if marginal costs are not equal.

15.8 This is an entry game with important first-mover advantages.

15.9 This is a game theory example from the theory of cartels. Because the stable price is so low, cartels may seek enforcement mechanisms to maintain higher (non-stable) prices.

15.10 This is an extension of Example 15.5. In this case, the firms must consider the expected value of profits when choosing trigger price strategies.

15.11 This problem provides a numerical example of Bayesian Nash equilibrium in which demand (rather than costs) is uncertain for player B.

15.12 This is a problem in auction theory. The results from Example 15.9 are extended and Vickrey's second-price auction is introduced. The mathematics here is a bit difficult, but the hints should help students through.

Solutions

15.1 a. Stag-Stag and Hare-Hare are both Nash equilibria.

b. Let p = Probability A plays stag. B's payoffs are

Stag $2p + 0(1 - p) = 2p$

Hare $p + (1 - p) = 1$

So Stag payoff > Hare if $2p > 1$ or if $p > 0.5$.

c. B's payoff to Stag with n players is $2p^{n-1}$ (since all must cooperate to catch a stag). Hence B will play stag if $2p^{n-1} > 1$ or if $p^{n-1} > 0.5$.

15.2 Payoffs are d_A , d_B if $d_A + d_B -$ 100, and 0 if $d_A + d_B > 100$. All strategies for which $d_A + d_B = 100$ represent Nash equilibria since no player has an incentive to change given the other player's strategy.

15.3 Using notation from Example 10.2, expected utility for A is

$$U_A = 1 - s + r\,[(K + 1)s - 1]$$

and for B

$$U_B = K(1 - r) + s\,[(K + 1)s - K].$$

Hence mixed strategy equilibrium is

$$s = 1/(K + 1) \qquad r = K/(K + 1).$$

15.4 This is solved through backward induction. Parent's maximum for L requires $-U'_B + \lambda U'_A = 0$

Child's maximum choice of r is $U'_A\,(Y'_A + dL/dr) = 0$

So $Y'_A + dL/dr = 0$

Differentiation of parent's optimum with respect to r yields $-U''_B(Y'_B - dL/dr) + \lambda U''_A(Y'_A + dL/dr) = 0$

but $Y'_A + dL/dr = 0$ from child maximum problem, so $-U''_B(Y'_B - dL/dr) = 0$ or $dL/dr = Y'_B$.

Hence, $Y'_A + Y'_B = 0$ which is precisely the condition required for r to maximize total income.

15.5 a. There are two Nash equilibria here:

A: Chicken, B: Not Chicken; and A: Not Chicken, B: Chicken.

b. The threat "Not Chicken"; is not credible against a firm commitment by one's opponent to Not Chicken.

c. Such a commitment would achieve a desirable result assuming the opponent has not made such a commitment also.

d. The film appears to have a group of men trying to decide who will approach an attractive woman. If they all do, all will be rejected. But going first for everyone is the Nash equilibrium. Hence, there needs to be some sort of pre-game decision process to choose the first mover.

15.6 a. Strategies here are continuous. $A = 500.01$ dominates any strategy in which A bids more. $B = 500$ dominates any strategy for which B bids more. Any other strategies are not dominant

b. The only Nash equilibrium here is $A = 500.01$, $B = 500$.

c. With imperfect information this becomes a Bayesian game. See Example 15.8 for a discussion.

15.7 a. Here B's optimal strategy is to choose a price slightly less than 10. This is a Nash equilibrium. With that price $q_A = 0$, $q_B = 300$.

b. $\pi_A = 0$, $\pi_B = 600$

c. This equilibrium is inefficient because $P > MC_B$. Efficient allocation would have $P = 8$, $q_B = 340$, $\pi_B = 0$.

15.8 a. This game has two Nash equilibria:

(1) A = Produce, B = Don't Produce, and

(2) A = Don't Produce, B = Produce.

b. If A moves first, it can dictate that Nash equilibrium (1) is chosen. Similarly, if B goes first, it can assure that Nash equilibrium (2) is chosen.

c. Firm B could offer a bribe of 1 to firm A not to enter (if it is A's move first). But this would yield identical profits to those obtained when A moves first anyway.

15.9 a. If owners act as a cartel, they will maximize total revenue $= P \cdot Q = 10{,}000P - 1{,}000P^2$

$$\frac{dPQ}{dP} = 10{,}000 - 2{,}000P.$$

Hence, $P = 5$ $\quad Q = 5{,}000$.

For each owner, $q = 250$.

Revenues per firm = 1250.

b. $P = 5$ is unstable since if one firm sells 251,

$Q = 5001 \quad P = 4.999$.

Revenue for the cheating firm = 1254.7 so chiseling increases revenues and profits for the single firm.

c. With a suitably low price, there will be no incentive to cheat. With $P = .30$, for example, $Q = 9700$ and $q = 485$. Revenues per firm = 145.50.

If $q = 486 \quad P = .299$.

Revenue for the cheating firm = 145.31 so there is no incentive to cheat.

Notice that with fewer cartel members, this stable price is higher. With 2 firms, for example, if $P = 3$, $Q = 7{,}000$ and $q = 3500$. Revenues per firm = 10,500.

If $q = 3501$, $P = 2.999$.

Revenue for the cheating firm = 10,499.50 so there is no incentive to chisel.

15.10 a. Monopoly price in expansions is $P = 40$, $\pi_e = 90{,}000$.

In recessions, $P = 20 \qquad \pi_r = 10{,}000$.

Hence, long-term expected profits from a trigger price strategy (under which price is set appropriately once demand conditions are known) are 50,000 per period. Sustainability requires that cheating during expansions be unprofitable

$$90{,}000 < \frac{50{,}000}{2}\,\frac{1}{1-\delta}$$

which holds for

$\delta > .72$.

b. Lower δs will permit sustainability providing profits during expansions (π_e) satisfy the condition

$$\pi_e < \frac{0.5\cdot(\pi_e + 10{,}000)}{2}\,\frac{1}{1-\delta}\,.$$

For $\delta = .7$ (say), $\pi_e < 50{,}000$ which requires the same price during expansions ($P = 20$) as recessions.

15.11 Again, best to start by analyzing B's situation. Denote the two demand situations by "1" and "2."

$$\pi_{B1} = (110 - q_A - q_B)q_B$$

$$\pi_{B2} = (70 - q_A - q_B)q_B\ .$$

Hence,

$$q^*_{B1} = (110 - q_A)/2$$

$$q^*_{B2} = (70 - q_A)/2 .$$

Now

$$\pi_A = .5(90 - q_A - q_{B1})q_A + .5(90 - q_A - q_{B2})q_a$$

$$= (90 - q_A - .5q_{B1} - .5q_{B2})q_A$$

so optimal q_A is

$$q^*_A = (90 - .5q_{B1} - .5q_{B2})/2 .$$

Solving the three optimal strategies simultaneously yields:

$$q^*_A = 30 \quad q^*_{B1} = 40 \quad q^*_{B2} = 20 .$$

15.12 a. The hint follows because the probability of a bid less than *v* is in fact given by *v* because of the assumed distribution of *v*. Hence, the probability that $n - 1$ bids will be lower than v is given by v^{n-1} and this can occur for any of the n bidders. Hence $f(v) = nv^{n-1}$.

$$E(v^*) = \int_0^1 nv^n dv = \frac{n}{n+1} v^{n+1} \Big|_0^1 = \frac{n}{n+1}$$

Hence, expected revenue is $\frac{n-1}{n} \cdot \frac{n}{n+1} = \frac{n-1}{n+1}$.

b. The Vickrey scheme is "truth revealing" because each bidder has the incentive to bid his/her true value. If that true value were the highest, bidding it would not affect what is paid which is determined by the second highest bidder. On the other hand, if it is not the highest, might as well bid it anyway because won't win the auction.

c. Given the hint,

$$E(v^*) = \int_0^1 (n-1)n(1-v)v^{n-1} dv = (n-1)n\left[\frac{v^n}{n}\Big|_0^1 - \frac{v^{n+1}}{n+1}\Big|_0^1\right]$$

$$= (n-1)n\left(\frac{1}{n} - \frac{1}{n+1}\right) = (n-1)n\left(\frac{1}{n(n+1)}\right) = \frac{n-1}{n+1} .$$

Notice that the difference in the auctions is not in the expected revenue, but in the fact that the Vickrey auction is truth revealing whereas the first bid auction is not.

CHAPTER 16

LABOR MARKETS

Because the subject of labor demand was treated extensively in Chapter 9, the problems in this chapter focus primarily on labor supply and on equilibrium in the labor market. Most of the labor supply problems (16.1–16.6) start with the specification of a utility function and then ask students to explore the labor supply behavior implied by the function. The primary focus of most of the problems that concern labor market equilibrium is on monopsony and the marginal expense concept (problems 16.7 – 16.10).

Comments on Problems

16.1 This is a simple algebraic example of labor supply that is based on a Cobb– Douglas (constant budget shares) utility function. Part (b) shows, in a simple context, the work disincentive effects of a lump-sum transfer—3/4 of the extra 4000 is "spent" on leisure which, at a price of $5 per hour implies a 600 hour reduction in labor supply. Part (c) then illustrates a positive labor supply response to a higher wage since the $3000 spent on leisure will now only buy 300 hours. Notice that a change in the wage would not affect the solution to part (a), because, in the absence of nonlabor income, the constant share assumption assures that the individual will always choose to consume 6000 hours (= 3/4 of 8000) of leisure.

16.2 A problem using the expenditure function approach to study labor supply. Shows why income and substitution effects are precisely offsetting in the Cobb-Douglas case.

16.3 A risk-aversion example that shows that wages must be higher on jobs with some uncertainty about the income stream promised if they are to yield the same utility as jobs with no uncertainty. The problem requires students to make use of the concepts of standard deviation and variance and will probably make little sense to students who are unfamiliar with those concepts.

16.4 A problem in family labor supply theory. Introduces (in part [b]) the concept of "home production." The functional forms specified here are so general that this problem should be regarded primarily as a descriptive one that provides students with a general framework for discussing various possibilities.

16.5 An application of labor supply theory to the case of means-tested income transfer programs. Results in a kinked budget constraint. Reducing the implicit tax rate on earnings (parts [f] and [g]) has an ambiguous effect on H since income and substitution effects work in opposite directions.

16.6 A simple supply-demand example that asks students to compute various equilibrium positions.

16.7 An illustration of marginal expense calculation. Also shows that imposition of a minimum wage may actually raise employment in the monopsony case.

16.8 An example of monopsonistic discrimination in hiring. Shows that wages are lower for the less elastic supplier. The calculations are relatively simple if students calculate marginal expense correctly.

16.9 A bilateral monopoly problem for an input (here, pelts). Students may get confused on what is required here, so they should be encouraged first to take an *a priori* graphical approach and then try to add numbers to their graph. In that way, they can identify the relevant intersections that require numerical solutions.

16.10 A numerical example of the union-employer game illustrated in Example 16.5.

Solutions

16.1 a. 8000 hrs/year @ $5/hr = $40,000/year

3/4 · $40,000/yr = $30,000/yr at leisure.

$\frac{\$30,000}{\$5}$ = 6,000 hours of leisure.

Work = 2,000 hours.

b. 3/4 · $44,000/yr = $33,000/yr at leisure.

$\frac{\$33,000}{\$5}$ = 6,600 hours of leisure.

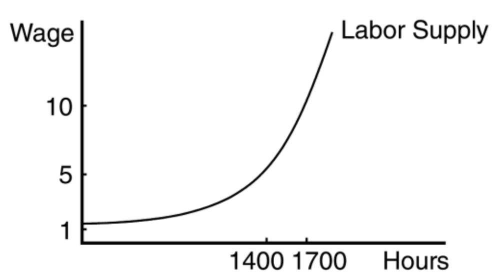
Work = 1,400 hours.

c. Now, full income = $84,000.

3/4 · $84,000 = $63,000.

Leisure = 6,300 hours; work = 1,700 hours. Hence, higher wage leads to more labor supply. Note that in part (a) labor supply is perfectly inelastic at 2,000 hours.

d.



16.2 a. Setting up Lagrangian: $? = c + wh - \ 4w + \lambda(\bar{U} - c^{\alpha}h^{1-\alpha})$ gives the following first order conditions:

$$\frac{\partial \pounds}{\partial c} = 1 - \lambda\alpha c^{\alpha-1}h^{1-\alpha} = 0$$

$$\frac{\partial \pounds}{\partial h} = w - \lambda(1-\alpha)c^{\alpha}h^{-\alpha} = 0$$

$$\frac{\partial \pounds}{\partial \lambda} = \bar{U} - c^{\alpha}h^{1-\alpha} = 0$$

Combining the first two equations gives the familiar result: $\frac{1}{w} = \frac{\alpha h}{(1-\alpha)c}$.

So $c = \frac{\alpha wh}{1-\alpha}$, and $\bar{U} = (\frac{\alpha}{1-\alpha})^{\alpha} w^{\alpha} h$ or $h = \bar{U}k^{-\alpha}w^{-\alpha}$ where $k = \frac{\alpha}{(1-\alpha)}$.

Similar substitutions shows $c = \bar{U}k^{1-\alpha}w^{1-\alpha}$.

Substituting for expenditures gives

$E = c + wh - 24w = \bar{U}w^{1-\alpha}[k^{-\alpha} + k^{1-\alpha}] - 24w = \bar{U}w^{1-\alpha}K - 24w$

Where $K = k^{-\alpha} + k^{1-\alpha}$.

b. $h^c = \frac{\partial E}{\partial w} = (1-\alpha)\bar{U}w^{-\alpha}K - 24$.

c. $l^c = 24 - h^c = 48 - (1-\alpha)\bar{U}w^{-\alpha}K$

Clearly $\frac{\partial l^c}{\partial w} = \alpha(1-\alpha)\bar{U}Kw^{-\alpha-1} > 0$.

d. The algebra is considerably simplified here by assuming $\alpha = 0.5, K = 2$ and using a period of 1.0 rather than 24. With these simplifications,

$l^c = 2 - \bar{U}w^{-0.5}$ $\quad l = 0.5 - 0.5nw^{-1}$. $\frac{\partial l^c}{\partial w} = 0.5\bar{U}w^{-1.5}$. Now letting $n = E$ in the expenditure function and solving for utility gives $\bar{U} = 0.5w^{0.5} + 0.5nw^{-0.5}$. Substituting gives $\frac{\partial l^c}{\partial w} = 0.25w^{-1}$ when $n = 0$. Turning to the uncompensated function: $l \cdot \frac{\partial l}{\partial n} = (0.5 - 0.5nw^{-1})(-0.5w^{-1}) = -0.25w^{-1}$ when $n = 0$. Hence, the substitution and income effects cancel out.

(Note: In working this problem it is important not to impose the $n = 0$ condition until after taking all derivatives.)

16.3 $U(Y) = 100Y - 0.5Y^2$ $\qquad Y = wl$

job: \$5, 8-hour day $Y = 40$, $U = 3200$

$E(U)_{\text{job}}$ = 3200; to take new job, $E(U)_{\text{job2}} > E(U)_{\text{job1}}$

$$E(U)_{job2} = E(100Y - 0.5Y^2) = 800w - 0.5E(Y^2)$$

$$= 800w - 0.5(\text{var}\, Y + [E(Y)]^2)$$

because $Y = lw$, $E(Y) = 8w$, $sd(Y) = 6w$

$$E(U)_{job2} = 800w - 0.5(36w^2 + 64w^2) \geq 3200$$

$50w^2 - 800w + 3200 \geq 0$. Use quadratic formula, get $w \geq 8$.

16.4 a. $\frac{\partial h_1}{\partial w_2}$ and $\frac{\partial h_2}{\partial w_1}$ are both probably positive because of the income effect.

b. $c_1 = f(h_1)$, so optimal choice would be to choose h_1 so that $f' = w_1$. This would probably lead person 1 to work less in the market. That may in turn lead person 2 to choose a lower level of h_2 on the assumption that h_1 and h_2 are substitutes in the utility function. If they were complements, the effect could go the other way. Clearly one can greatly elaborate on this theory by working out all of the first-order conditions and comparative statics results.

16.5 a. Grant = 6000 – .75(I)

If I = 0	Grant = 6000
I = 2000	Grant = 4500
I = 4000	Grant = 3000.

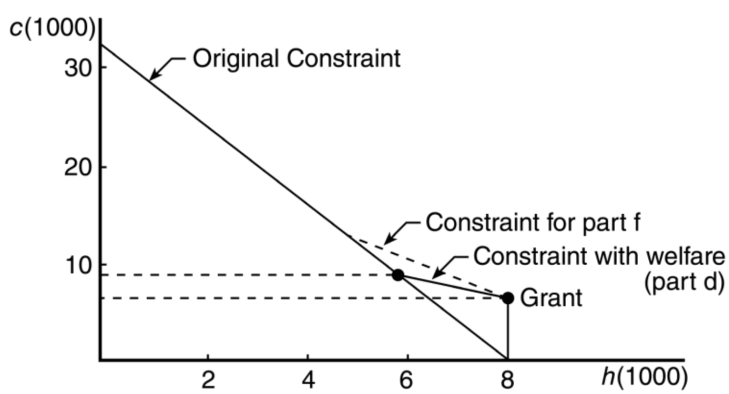
b. Grant = 0 when 6000 – .75I = 0

I = 6000/.75 = 8000

c. Assume there are 8000 hours in the year.

Full Income = 4 • 8000 = 32,000 = $c + 4h$.

d. Full Income = 32,000 + grant = 32,000 + 6000 – .75 · 4(8000 – h) = 38,000 – 24,000 + 3h = $c + 4h$ or 14,000 = $c + h$ for I – – 8,000. That is: for h > 6,000 hours welfare grant creates a kink at 6,000 hours of leisure.

e. 

f. New budget constraint is $23{,}000 = c + 2h$ for $h > 5{,}000$.

g. Income and substitution effects of law change work in opposite directions (see graph). Substitution effect favors more work; income effect, less work.

16.6 D: $L = -50w + 450$ S: $L = 100w$

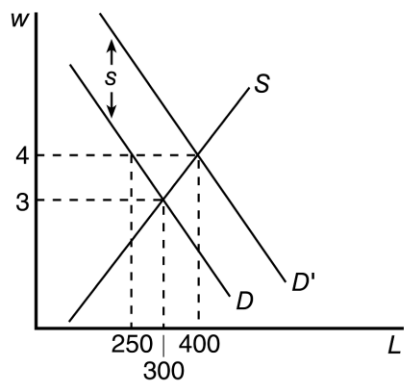
a. $S = D$ $100w = -50w + 450$ $w = 3$, $L = 300$

b. D: $L = -50(w - s) + 450$ s = subsidy

$w = 4$ $L_s = 400 = -50(4 - s) + 450$ $s = 3$

Total subsidy is 1200.

c. $w = \$4$ $D = 250$ $S = 400$ $u = 150$



d.

16.7 Supply: $l = 80w$ $ME_l = \frac{l}{40}$ Demand: $l = 400 - 40MRP_l$

a. For monopsonist $ME_l = MRP_l$

$l = 400 - 40MRP_l$ $MRP_l = 10 - \frac{l}{40}$

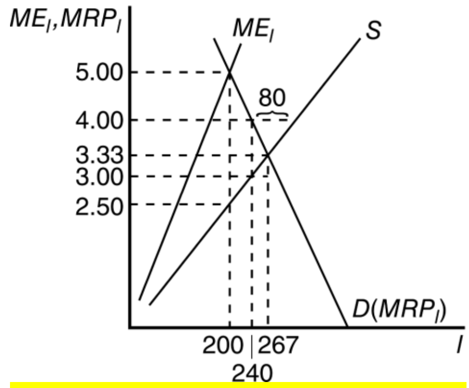
$\frac{l}{40} = 10 - \frac{l}{40}$ $l = 200$

Get w from supply curve. $w = \frac{l}{80} = \frac{200}{80} = 2.50$.

b. For Carl, the marginal expense of labor now equals the minimum wage, and in equilibrium the marginal expense of labor will equal the marginal revenue product of labor. $w_m = ME_l = MRP_l$

	Demand	Supply
$w_m = \$4.00$	$l = 400 - 40(4.00)$	$l = 80(4.00)$
	$l = 240$	$l = 320$

If supply > demand, Carl will hire 240 workers, unemployment = 80.

c. 

d. Under perfect competition, a minimum wage means higher wages but fewer workers employed. Under monopsony, a minimum wage may result in higher wages *and* more workers employed.

16.8 $w_m^2 = \frac{l_m}{9} \quad w_m l_m = \frac{l_m^{3/2}}{3}$

$ME_l = \frac{l_m^{0.5}}{2} = MRP_l = 10$ so $\quad l_m = 400, w_m = \frac{20}{3}$

$w_f = \frac{l_f}{100} \quad w_f \; l_f = \frac{l_f^2}{100}$

$ME_l = \frac{l_f}{50} = 10$ so $\quad l_f = 500, w_f = 5 \quad l_T = 900$

profits per hour on machinery = 9000 – 5(500) – 6.66(400) = 3833.

If same wage for men and women $w = MRP_l = 10$, $l = 1000 + 900 = 1900$.

Profits per hour are now = 1900(10) – 10(1000) – 10(900) = 0.

16.9 a. Since $q = 240x - 2x^2$, $R = 5q = 1200x - 10x^2$

MRP for pelts = $\frac{\partial R}{\partial x} = 1200 - 20x$.

Production of pelts $x = \sqrt{l} \quad C = wl = 10x^2, \quad MC = 20x$.

Under competition, price of pelts $p_x = MC = 20x$ and $MRP_x = p_x$

$x = 30 \quad p_x = 600$.

b. From Dan’s perspective, demand for pelts $= MRP_x = 1200 - 20x$

$R = p_x \cdot x = 1200x - 20x^2$. $MR = \frac{\partial R}{\partial x} = 1200 - 40x$

Set $MR = MC = 20x \;\; x = 20 \quad p_x = 800$.

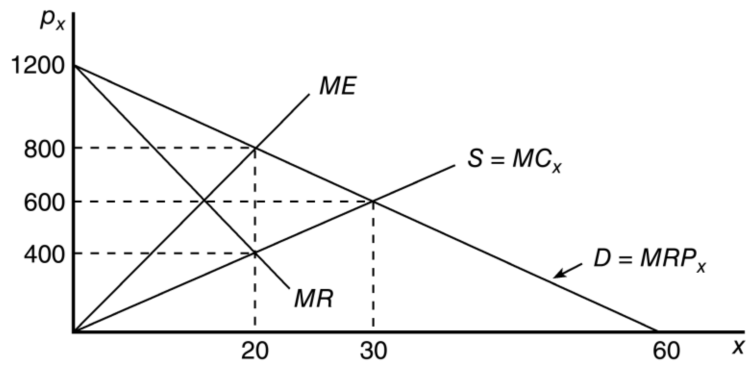
c. From *UF*'s perspective supply of pelts = $MC = 20x = p_x$. total cost = $p_x x = 20x^2$.

$$ME_x = \frac{\partial C}{\partial x} = 40x \ .$$

Profit maximization set $ME_x = 40x = MRP_x = 1200 - 20x$

$x = 20 \quad p_x = 400.$

d. Both the monopolist and monopsonist agree on $x = 20$, but they differ widely on price to be paid. Bargaining will determine the result.



16.10 a. As in Example 16.5, this is solved by backward induction. In the second stage of the game the employer chooses *l* to maximize $10l - l^2 - wl$.

which requires $l = 5 - w/2$.

Union chooses *w* to maximize $wl = 5w - 0.5w^2$

so $w^* = 5$, $l^* = 2.5$ $U^* = 12.5$, $\pi^* = 6.25$.

b. With $w' = 4, l' = 4, U' = 16, \pi' = 8.$ which is Pareto-superior to the contract in part a.

c. For sustainability, one needs to focus on the employer who has incentive to cheat if union chooses $w' = 4$ (profit maximizing *l* is 3, not 4). Since $\pi(l = 3) = 9$, the condition for sustainability is $8/(1 - \delta) > 9 + 6.25\delta/1 - \delta)$ or $\delta > 1/2.75 = 4/11$.

CHAPTER 17

CAPITAL MARKETS

The problems in this chapter are of two general types: (1) those that focus on intertemporal utility maximization and (2) those that ask students to make present discounted value calculations. Before undertaking the *PDV* problems, students should be sure to read the Appendix to Chapter 17. That appendix is especially important for problems involving continuous compounding because students may not have encountered that concept before. Because the material on dynamic optimization is rather difficult, only one problem on it is included (17.10).

Comments on Problems

17.1 A graphic analysis of intertemporal choices. Illustrates the indeterminacy of the sign of the interest elasticity of current savings. Part (c) concerns intertemporal allocation with initial endowments in both periods.

17.2 A present discounted value problem. I have found that the problem is most easily solved using continuous compounding (see below), but the discrete approach is also relatively simple. Instructors may wish to point out that the savings rate calculated here (22.5 percent) is considerably above the personal savings rate in the United States. That could lead into a discussion of the possible effects of social security.

17.3 A simple present discounted value problem that should be solved with continuous compounding.

17.4 A traditional capital theory problem. Students seem to have difficulty in seeing their way through this problem and in interpreting the results. Hence, instructors may wish to allow some time for discussion of it.

17.5 Further analysis of forestry economics shows how replanting costs affect *PDV* calculations.

17.6 A discussion question that asks students to explore the logic of the U.S. corporate income tax. The case of accelerated depreciation is, I believe, a particularly effective example of the time value of money.

17.7 A present discounted value example of life insurance sales tactics. Students tend to like this problem and, I'm told, some have even used its results when approached by actual salespeople.

17.8 An intertemporal resource allocation example of the capital gains that arise from taxation of the capital gains from interest rate changes.

17.9 A resource economics problem that shows, with a finite resource, monopoly pricing options are severely constrained.

17.10 A simple application of control theory to optimal savings decisions. Provides an alternative derivation of the "Euler Equation." Final parts of the problem illustrate the significance of intertemporal substitutability.

Solutions

17.1 a. $? = U\ c_1, c_2) + \lambda\left(W - c_1 - \frac{c_2}{1+r}\right)$

$$\frac{\partial\ £}{\partial c_1} = \frac{\partial U}{\partial c_1} - \lambda = 0$$

$$\frac{\partial\ £}{\partial c_2} = \frac{\partial U}{\partial c_2} - \frac{\lambda}{1+r} = 0$$

Division of these first order conditions yields $\dfrac{\frac{\partial U}{\partial c_1}}{\frac{\partial U}{\partial c_2}} = 1 + r = MRS$

b. $\frac{\partial c_2}{\partial r} \geq 0$ because c_2 is a normal good with price $\frac{1}{1+r}$. $\frac{\partial c_1}{\partial r}$ is ambiguous because the substitution effect predicts $\frac{\partial c_1}{\partial r} < 0$ but the income effect predicts $\frac{\partial c_1}{\partial r} > 0$. If $\frac{\partial c_1}{\partial \text{r}} < 0$, a fall in the price of c_2 raises total spending on c_2 raises total spending on c_2. Therefore, demand for c_2 is elastic.

c. Budget constraint has same slope as in part (a) and passes through $c_1 = y_1$, $c_2 = y_2$. If at optimal point $c_1^* > y_1$, the individual borrows in period 1 and repays in period 2. If $c_1^* < y_1$ individual saves in period 1 and uses savings in period 2.

17.2 Use continuous time for simplicity.

$y_t = y_0 e^{.03t} \qquad y_{40} = y_0 e^{1.2} \qquad s_t = sy_t = sy_0 e^{.03t}$

Accumulated savings after 40 years

$$= \int_0^{40} s_t e^{.03(40-t)}\,dt = sy_0 \int_0^{40} e^{.03t}\ e^{.03(40-t)}\ dt = sy_0 \int_0^{40} e^{1.2}\,dt = sy_0\,40e^{1.2}$$

Present value of spending in retirement

$$= \int_0^{20} .6 y_{40}\, e^{-.03t}\, dt = .6 y_0 e^{1.2} \int_0^{20} e^{-.03t}\, dt = .6 y_0 e^{1.2} \left. \frac{e^{-.03t}}{-.03} \right|_0^{20} = .6 y_0 e^{1.2} (15.04)$$

For accumulated savings to equal *PDV* of dissavings, it must be the case that

$$s = \frac{.6 y_0 e^{1.2} (15.04)}{40 y_0 e^{1.2}} = \frac{9}{40} = 0.225$$

17.3 $$.05 = \frac{f'(t)}{f(t)} = \frac{(t^{-0.5} - .15)\ (e^{2\sqrt{t} - .15t})}{e^{2\sqrt{t} - .15t}}$$

$.05 = t^{-0.5} - .15$ $\quad t^{-0.5} = .2 \quad t = 25$ years

17.4 a. $PDV = e^{-rt} f(t)$

$$\frac{dPDV}{dt} = e^{-rt}\ f'(t) - f(t)(re^{-rt}) = 0$$

$f'(t) - rf(t) = 0 \qquad r = f'(t)/f(t)$ at t^*.

Since w paid currently, $\pi = 0$ requires $w = e^{-rt} f(t^*)$

b. Value of a u year-old tree: $= e^{-r(t-u)} f(t^*) = we^{ru}$

we^{ru} grows at rate r, tree grows faster than r except at t^*.

we^{ru} starts out above $f(t)$ and $f(t)$ catches up at t^*.

c,d. $$V = \int_0^{t^*} we^{ru}\, du = w \int_0^{t^*} e^{ru} du = w \frac{e^{ru}}{r} \Big|_0^{t^*} = \left[we^{rt^*} - w \right] \frac{1}{r}$$

So $rV = f(t^*) - w$

17.5 a. Since $\dfrac{x}{1-x} = x + x^2 + \ldots$ for $x < 1$

$$V = -w + [f(t) - w]\ [e^{-rt}/(1 - e^{-rt})].$$

$$= -w + [f(t) - w]/(e^{rt} - 1)$$

b. $$dV/dt = \frac{(e^{rt} - 1)\, f'(t) - [f(t) - w]\, re^{rt}}{(e^{rt} - 1)^2}$$

So, for a maximum,

$$0 = \frac{f'(t)}{e^{rt} - 1} - \frac{[f(t) - w]\, re^{rt}}{(e^{rt} - 1)^2}$$

$$f'(t) = \frac{[f(t) - w]re^{rt}}{e^{rt} - 1}$$

$$= \frac{f(t) - w]re^{rt}}{e^{rt} - 1} + \frac{r[f(t) - w] - r[f(t) - w]}{e^{rt} - 1}$$

$$= \frac{rf(t)[e^{rt} - 1]}{[e^{rt} - 1]} + \frac{r[f(t) - w]}{[e^{rt} - 1]} - \frac{rw[e^{rt} - 1]}{[e^{rt} - 1]}$$

$$= rf(t) + rV(t).$$

c. The condition implies that, at optimal t^*, the increased wood obtainable from lengthening t must be balanced by: (1) the delay in getting the first rotation's yield; and (2) the opportunity cost of a one-period delay in all future rotations' yield.

d. $f(t)$ is asymptotic to 50 as $t \to \infty$.

e. $t^* = 100$ years. This is not "maximum yield" since tree always grows.

f. Now $t^* = 104.1$ years. Lower r lengthens the growing period.

17.6 a. Not at all, because there are no pure economic profits in the long run.

b. In long-run equilibrium: $v = P_K(r + d)$. Government taxes opportunity cost of capital. This raises v and provides an incentive to substitute labor for capital.

c. Tend to increase use of capital since there is a tax advantage in early years. Total taxes paid are equal, but timing of payments is different. Consequently, present value of tax liabilities under accelerated depreciation is less than under straight line.

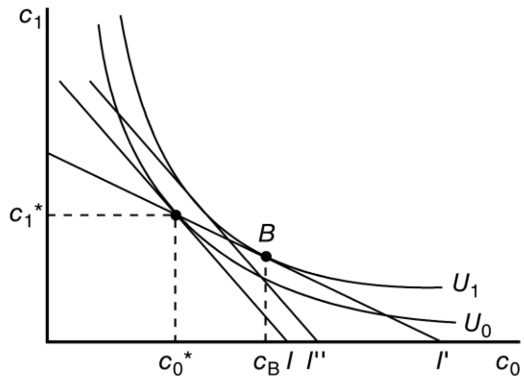
d. If tax rate declines, tax benefits of accelerated depreciation are smaller. May reduce investment.

17.7 $$PDV_{\text{whole life}} = \int_0^4 2{,}000e^{-.1t}\,dt = \$6{,}304$$

$$PDV_{\text{term}} = \int_0^{35} 400e^{-.1t}\,dt = \$3{,}879.$$

The salesman is wrong. The term policy represents a better value to this consumer.

17.8



a. Current savings = $I - c_0^*$.

b. Once one-period bonds are purchased, fall in r causes budget constraint to rotate to I'. Increase in utility from U_0 to U_1 (point B) represents a capital gain.

c. Accrued capital gains are measured by the total increase in ability to consume c_0 (this is the "Haig-Simmons" definition of income)—measured by distance II'.

d. Realized capital gains are given by distance $c_0^* c_B$ that is the present value of one-period bonds that must be sold to attain the new utility-maximizing choice of c_B.

e. The "true" capital gain is given by the value, in terms of c_0, of the utility gain. That is measured by II''. Notice that this is smaller than either of the "gains" calculated in parts (c) or (d). Hence, the current practice of taxing realized gains, while more appropriate than full taxation of all accrued gains, still amounts to some degree of over-taxation because it neglects effects on c_1 consumption opportunities.

17.9 Final P_n implies a final $MR_n = p_n(1 + 1/k)$ (where k is the elasticity of demand for oil). Logic of resource theory suggests MR must grow at rate r. Hence

$MR_0 = MR_n e^{-rn} = p_n(1 + 1/k)e^{-rn}$. If k is constant over time, this implies

$p_0 = MR_0 /(1 + 1/k) = p_n e^{-rn}$, so competitive pricing must prevail.

17.10 a. The augmented Hamiltonian for this problem is

$$H = U(c)e^{-\rho t} + \lambda(w + rk - c) + k\dot{\lambda} .$$

Differentiation with respect to c yields

$$\partial H / \partial c = U'(c)e^{-\rho t} - \lambda = 0 .$$

Differentiation with respect to k yields

$$\partial H / \partial k = r\lambda + \dot{\lambda} = 0 .$$

Hence, $\lambda = e^{-rt}$ and $U'(c)e^{-\rho t} = e^{-rt}$ $U'(c) = e^{-(r-\rho)t}$

b. From (a), if $r = \rho$, c is constant. If $r > \rho$, U' must fall as t increases, so c must rise. Alternatively, if $r < \rho$, U' must rise over time, so c must fall.

c. If $U(c) = \ln c$ $U' = 1/c$

so, $c = c_0 e^{(r-\rho)t}$; that is, c follows an exponential path (either rising or falling).

d. If $U(c) = c^{\delta}/\delta$, $U'(c) = c^{\delta-1}$ so $c^{\delta-1} = c_0 e^{-(r-\rho)t}$ or $c = c_0 e^{-[(r-\rho)/(\delta-1)]t}$
Because $\delta < 1$, this gives the same qualitative predictions, but the growth rate (assuming $r > \rho$) of consumption now depends on δ too. The less substitutable are various periods' consumption (the more negative is δ), the slower will be consumption growth. If $\delta = -\infty$ (no substitution), c is a constant, as in the $r = \rho$ case.

e. Because income is constant over this life cycle, wealth will be determined solely by consumption patterns. It will have a humped shape if consumption rises over time, but wealth will be negative if consumption falls over time (notice that the budget constraint here allows unlimited borrowing).

CHAPTER 18

UNCERTAINTY AND RISK AVERSION

Most of the problems in this chapter focus on illustrating the concept of risk aversion. That is, they assume that individuals have concave utility of wealth functions and therefore dislike variance in their wealth. A difficulty with this focus is that, in general, students will not have been exposed to the statistical concepts of a random variable and its moments (mean, variance, etc.). Most of the problems here do not assume such knowledge, but the Extensions do show how understanding statistical concepts is crucial to reading applications on this topic.

Comments on Problems

18.1 Reverses the risk-aversion logic to show that observed behavior can be used to place bounds on subjective probability estimates.

18.2 This problem provides a graphical introduction to the idea of risk-taking behavior. The Friedman-Savage analysis of coexisting insurance purchases and gambling could be presented here.

18.3 This is a nice, homey problem about diversification. Can be done graphically although instructors could introduce variances into the problem if desired.

18.4 A graphical introduction to the economics of health insurance that examines cost-sharing provisions. The problem is extended in Problem 19.3.

18.5 Problem provides some simple numerical calculations involving risk aversion and insurance. The problem is extended to consider moral hazard in Problem 19.2.

18.6 This is a rather difficult problem as written. It can be simplified by using a particular utility function (e.g., $U(W) = \ln W$). With the logarithmic utility function, one cannot use the Taylor approximation until after differentiation, however. If the approximation is applied before differentiation, concavity (and risk aversion) is lost. This problem can, with specific numbers, also be done graphically, if desired. The notion that fines are more effective can be contrasted with the criminologist's view that apprehension of law-breakers is more effective and some shortcomings of the economic argument (i.e., no disutility from apprehension) might be mentioned.

18.7 This is another illustration of diversification. Also shows how insurance provisions can affect diversification.

18.8 This problem stresses the close connection between the relative risk-aversion parameter and the elasticity of substitution. It is a good problem for building an intuitive understanding of risk-aversion in the state preference model. Part d uses the CRRA utility function to examine the "equity-premium puzzle."

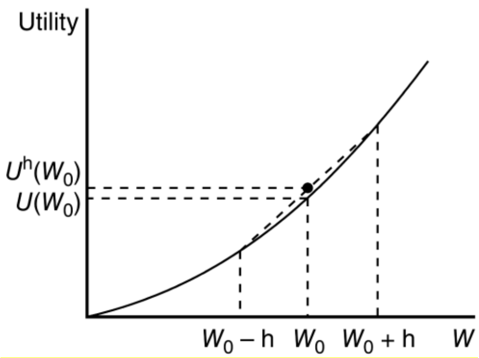
18.9 Provides an illustration of investment theory in the state preference framework.

18.10 A continuation of Problem 18.9 that analyzes the effect of taxation on risk-taking behavior.

Solutions

18.1 *p* must be large enough so that expected utility with bet is greater than or equal to that without bet: $p \ln(1{,}100{,}000) + (1 - p)\ln(900{,}000) > \ln(1{,}000{,}000)13.9108p + 13.7102(1 - p) > 13.8155$, $.2006p > .1053$ $p > .525$

18.2

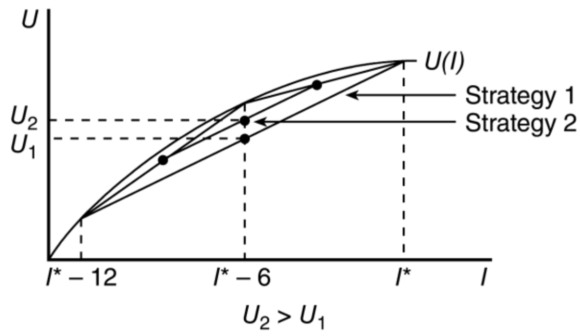


This would be limited by the individual's resources: he or she could run out of wealth since unfair bets are continually being accepted.

18.3 a.

	Outcome	Probability
Strategy One		
	12 Eggs	.5
	0 Eggs	.5
Expected Value =	.5•12 + .5•0 = 6	
Strategy Two	Outcome	Probability
	12 Eggs	.25
	6 Eggs	.5
	0 Eggs	.25
Expected Value	= .25 •12 + .5 • 6 + .25• 0	
	= 3 + 3 = 6	

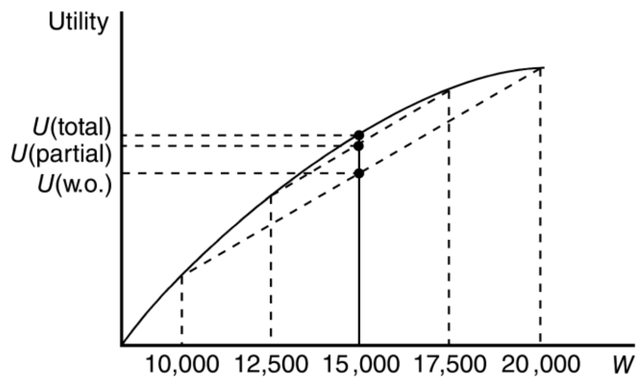
b.



18.4 a. $E(L)$ = .50(10,000) = \$5,000, so

Wealth = \$15,000 with insurance, \$10,000 or \$20,000 without.

b. Cost of policy is .5(5000) = 2500. Hence, wealth is 17,500 with no illness, 12,500 with the illness.



18.5 a. $E(U)$ = .75ln(10,000) + .25ln(9,000) = 9.1840

b. $E(U)$ = ln(9,750) = 9.1850

Insurance is preferable.

c. ln(10,000 – p) = 9.1840

10,000 – $p = e^{9.1840}$ = 9,740

p = 260

18.6 Expected utility = $pU(W - f) + (1 - p)U(W)$.

$$e_{U,p} = \frac{\partial U}{\partial p} \cdot \frac{p}{U} = [U(W - f) - U(W)] \cdot p / U$$

$$e_{U,f} = \frac{\partial U}{\partial f} \cdot \frac{f}{U} = -p \cdot U'(W - f) \cdot f / U$$

$$\frac{e_{U,p}}{e_{U,f}} = \frac{U(W-f) - U(W)}{-f\ U'(W-f)} < 1$$ by Taylor expansion,

So, fine is more effective.

If $U(W) = \ln W$ then Expected Utility = $p \ln (W - f) + (1 - p) \ln W$.

$$e_{U,p} = [\ln (W-f) - \ln\ W\ ] \cdot \frac{p}{U} \approx \frac{-p\ f\ /W}{U}$$

$$e_{U,f} = -\ p/U\ \ (W-f) \cdot \frac{f}{U} = \frac{-p\ f\ \ /(W-f)}{U}$$

$$\frac{e_{U,p}}{e_{U,f}} = \frac{W-f}{W} < 1$$

18.7 a. U (wheat) = .5 ln(28,000) + .5 ln(10,000) = 9.7251

U (corn) = .5 ln(19,000) + .5 ln(15,000) = 9.7340

Plant corn.

b. With half in each

Y_{NR} = 23,500 Y_R = 12,500

U = .5 ln(23,500) + .5 ln(12,500) = 9.7491

Should plant a mixed crop. Diversification yields an increased variance relative to corn only, but takes advantage of wheat's high yield.

c. Let α = percent in wheat.

U = .5 ln[(28,000) + (1 – α)(19,000)] + .5 ln[α (10,000) + (1 – α)(15,000)] = .5 ln(19,000 + 9,000α) + .5 ln(15,000 – 5,000α)

$$\frac{dU}{d\alpha} = \frac{4500}{19,000 + 9,000\alpha} - \frac{2500}{15,000 - 5,000\alpha} = 0$$

45(150 – 50α) = 25(190 + 90α) α = .444

U = .5 ln(22,996) + .5 ln(12,780) = 9.7494.

This is a slight improvement over the 50-50 mix.

d. If the farmer plants only wheat,

Y_{NR} = 24,000 Y_R = 14,000

U = .5 ln(24,000) + .5 ln(14,000) = 9.8163

so availability of this insurance will cause the farmer to forego diversification.

18.8 a. A high value for $1 - R$ implies a low elasticity of substitution between states of the world. A very risk-averse individual is not willing to make trades away from the certainty line except at very favorable terms.

b. $R = 1$ implies the individual is risk-neutral. The elasticity of substitution between wealth in various states of the world is infinite. Indifference curves are linear with slopes of –1. If $R = -\infty$, then the individual has an infinite relative risk-aversion parameter. His or her indifference curves are L-shaped implying an unwillingness to trade away from the certainty line at any price.

c. A rise in p_b rotates the budget constraint counterclockwise about the W_g intercept. Both substitution and income effects cause W_b to fall. There is a substitution effect favoring an increase in W_g but an income effect favoring a decline. The substitution effect will be larger the larger is the elasticity of substitution between states (the smaller is the degree of risk-aversion).

d.

i. Need to find R that solves the equation:

$$(W_0)^R = 0.5(1.055W_0)^R + 0.5(0.955W_0)^R$$

This yields an approximate value for R of –3, a number consistent with some empirical studies.

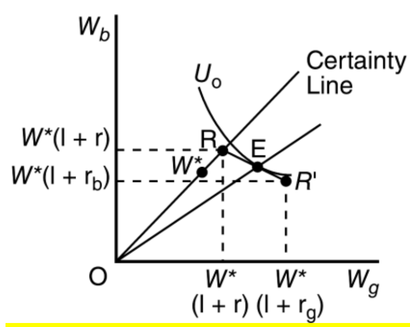
ii. A 2 percent premium roughly compensates for a ±10 percent gamble.

That is:

$$(W_0)^{-3} \approx (.92W_0)^{-3} + (1.12W_0)^{-3}.$$

The “puzzle” is that the premium rate of return provided by equities seems to be much higher than this.

18.9 a. See graph.



Risk free option is R, risk option is R'.

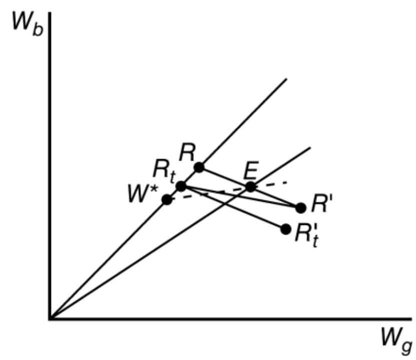
b. Locus RR' represents mixed portfolios.

c. Risk-aversion as represented by curvature of indifference curves will determine equilibrium in RR' (say E).

d. With constant relative risk-aversion, indifference curve map is homothetic so locus of optimal points for changing values of *W* will be along *OE*.

18.10 a. Because of homothetic indifference map, a wealth tax will cause movement along *OE* (see Problem 18.9).

b. A tax on risk-free assets shifts *R* inward to R_t (see figure below). A flatter $R_t\ R'$ provides incentives to increase proportion of wealth held in risk assets, especially for individuals with lower relative risk-aversion parameters. Still, as the "note" implies, it is important to differentiate between the after tax optimum and the before tax choices that yield that optimum. In the figure below, the no-tax choice is *E* on *RR'*. $W^* E$ represents the locus of points along which the fraction of wealth held in risky assets is constant. With the constraint $R_t\ R'$ choices are even more likely to be to the right of $W^* E$ implying greater investment in risky assets.



c. With a tax on both assets, budget constraint shifts in a parallel way to $R_t\ R_{t'}$. Even in this case (with constant relative risk aversion) the proportion of wealth devoted to risky assets will increase since the new optimum will lie along *OE* whereas a constant proportion of risky asset holding lies along $W_O\ E$.

CHAPTER 19

THE ECONOMICS OF INFORMATION

The problems in this chapter stress the economic value of information and illustrate some of the consequences of imperfect information. Only a few of the problems involve complex calculations or utilize calculus maximization techniques. Rather, the problems are intended primarily to help clarify the conceptual material in the chapter.

Comments on Problems

19.1 This problem illustrates the economic value of information and how that value is reduced if information is imperfect.

19.2 This is a continuation of Problem 18.5 that illustrates moral hazard and why its existence may prompt individuals to forego insurance.

19.3 Another illustration of moral hazard and how it might be controlled through cost-sharing provisions in insurance contracts.

19.4 This is an illustration of adverse selection in insurance markets. It can serve as a nice introduction to the topic of optimal risk classifications and to some of the economic and ethical problems involved in developing such classifications.

19.5 This is a simple illustration of signaling in labor markets. It shows that differential signaling costs are essential to maintaining a separating equilibrium.

19.6 An illustration of the economic value of price information. Notice that the utility of owning the TV is already incorporated into the function $U(Y)$ so all Molly wants to do is minimize the TV's cost.

19.7 A continuation of Problem 19.6 which uses material from the extensions to calculate the optimal number of stores to search.

19.8 A further continuation of Problems 19.6 and 19.7 that involves computation of an optimal reservation price.

19.9 This problem illustrates that principal-agent distortions may occur in medical care even when the physician is a "perfect" altruist.

19.10 Introduces the notion of "resolution-seeking" behavior. Here the notation is rather cumbersome (see the solutions for clarification).

Solutions

19.1 a. Expected profits with no watering are .5(1,000) + .5(500) = $750. With watering, profits are $900 with certainty. The farmer should water.

b. If the farmer knew the weather with certainty, profits would be $1,000 with rain, $900 with no rain. Expected profits are $950. The farmer would pay up to $50 for the information.

c. There are four possible outcomes with the following probabilities:

		Forecast	
		Rain	No Rain
Weather	Rain	37.5	12.5
	No Rain	12.5	37.5

Profits in each case are (assuming farmer follows forecaster's advice):

		Forecast	
		Rain	No Rain
Weather	Rain	1000	900
	No Rain	500	900

Expected profits, therefore, are

.375(1000) + .125(900) + .125(500) + .375(900) = 887.5.

The forecaster's advice is therefore of negative value to the farmer relative to the strategy of planning on no rain.

19.2 Premium is now $300. If she buys insurance, spending is 9700, utility = ln(9700) = 9.1799. This falls short of utility without insurance (9.1840), so here it is better to forego insurance in the presence of moral hazard.

19.3 A cost-sharing policy would now cost $1,750. Wealth when sick would be 20,000 – 1,750 – 3,500 = 14,750. Wealth when well would be 20,000 – 1,750 = 18,250. Utility from this combination may exceed utility of a certain $15,000.

19.4 a. Premium = (.8)(.5)(1,000) + (.2)(.5)(1,000) = 500

b. For *blue* without insurance

$E(U)$ = .8 ln9,000 + .2 ln10,000 = 9.1261.

With insurance

$E(U) = \ln(9{,}500) = 9.1590.$

Will buy insurance.

For *brown* without insurance

$E(U) = .2 \ln(9{,}000) + .8 \ln(10{,}000) = 9.1893.$

Better off without insurance.

c. Since only blue buy insurance, fair premium is 800.

Still pays this group to buy insurance.

[$E(U) = 9.1269$]

Brown will still opt for no insurance.

d. Blue premium = 800 $E(U) = 9.1269$

Brown premium = 200 $E(U) = 9.1901$

So Brown is better off under a policy that allows separate rate setting.

19.5 a. No separating equilibrium is possible since low-ability workers would always opt to purchase the educational signal identifying them as high-ability workers providing education costs less than $20,000. If education costs more than $20,000, no one would buy it.

b. A high-ability worker would pay up to $20,000 for a diploma. It must cost a low-ability worker more than that to provide no incentive for him or her to buy it too.

19.6 a. $U(18{,}000) = 9.7981$

b. $U(18{,}300) = 9.8147$

c. Utility of Trip = $.5U(18{,}200) + .5U(17{,}900) = 9.8009$. So since expected utility from the trip exceeds the utility of buying from the known location, she will make the trip.

19.7 a. Here $f(p) = \frac{1}{100}$ *for* $300 \le p \le 400$ and $f(p) = 0$ otherwise.

Cumulative function is $F(p) = \int_{300}^{p} f(x)dx = \frac{x}{100}\Big|_{300}^{p} = \frac{p}{100} - 3$

For $300 \le p \le 400$

$F(p) = 0$ for $P < 300$

$F(p) = 1$ for $P > 400$.

Expected minimum price (see footnote 1 of extension) is

$$p_{\min}^n = \int_0^{300} 1\ dp + \int_{300}^{400} \left(4 - \frac{p}{100}\right)^n dp$$

$$= 300 + \frac{-100}{(n+1)} \left(4 - \frac{p}{100}\right)^{n+1} \Big|_{300}^{400}$$

$$= 300 + \frac{100}{n+1}$$

b. $p_{\min}^n$ clearly diminishes with n:

$$d\,p_{\min}^n / dn = -100\,(n+1)^{-2} < 0$$

$$d^2 p_{\min}^n / dn^2 = 200\,(n+1)^{-3} > 0\,.$$

c. Set $dp_{\min}^n \ / dn = -100(n+1)^{-2} = -2$

$(n+1)^2 = 50$

n = 6.07 (i.e., 7 calls)

An intuitive analysis is:

With n = 6 $p_{\min}^n = 316.67$

With n = 7 $p_{\min}^n$ = 314.29

With n = 8 $p_{\min}^n = 312.50$

So should stop at the 7th call.

19.8 According to the Extensions, the searcher should choose p_R so that

$$C = 2 = \int_0^{p_R} F\ (p)\ dp$$

$$= \int_{300}^{p_R} \left(\frac{p}{100} - 3\right) dp = 50\left(\frac{p}{100} - 3\right)^2 \Big|_{300}^{p_R}$$

$$= 50\left(\frac{p_R}{100} - 3\right)^2 so\text{:}\ \ p_R = 320\ .$$

19.9 Patient utility maximization requires: $U_1^c / U_2^c = p_m$. Doctor Optimization requires: $U_1^d p_m + U_2^d [U_1^c - p_m U_2^c] = 0$. If $U_2^d = 1$ (which I interpret as meaning that the physician is a perfect altruist), this requires $p_m = U_1^c / (U_2^c - U_1^d)$. Relative to patient maximization, this requires a smaller U_1^c. Hence, the doctor chooses more medical care than would a fully informed consumer.

19.10 a. Expected value of utility = .5(10) + .5(5) = 7.5 regardless of when coin is flipped.

b. If coin is flipped before day 1, there is no uncertainty at day 2. From the perspective of day 1, utility = 10 or 5 with p = 0.5 so $E_1(U)$ = .5(10) + .5(5) = 7.5.

If the coin is flipped at day two, $E_2(U)$ = 7.5 and $E_1[E_2(U)]^1$ = 7.5 so date of flip does not matter.

c. With $\alpha = 2$, flipping at day 1 yields 100 or 25 with $p = 0.5$

$E_1(U)$ = .5(100) + .5(25) = 62.5.

Flipping at day 2 yields

$E_2(U)$ = .5(10) + .5(5) = 7.5 and $[E_2(U)]^2 = 56.25 < E_1(U)$.

Hence the individual prefers flipping at day 1.

d. With $\alpha = .5$, flipping at day 1 yields utility of $\sqrt{10}$ or $\sqrt{5}$ with p = 0.5

$E_1(U)$ = 2.70.

Flipping at day 2 yields $E_2(U)$ = .5(10) + .5(5) = 7.5 and $[E_2(U)]^{.5}$ = 2.74. Hence, the individual prefers flipping at day 2.

e. Utility is concave in c_2, but expected utility is linear in utility outcomes if $\alpha = 1$. Timing doesn't matter.

With $\alpha \neq 1$, timing matters because utility values themselves are exponentiated with a day-1 flip, whereas expected utility values are exponentiated with a day-2 flip. Values of $\alpha > 1$ favor a day-1 flip; values of $\alpha < 1$ favor a day-2 flip.

CHAPTER 20

EXTERNALITIES AND PUBLIC GOODS

The problems in this chapter illustrate how externalities in consumption or production can affect the optimal allocation of resources and, in some cases, describe the remedial action that may be appropriate. Many of the problems have specific, numerical solutions, but a few (20.4 and 20.5) are essay-type questions that require extended discussion and, perhaps, some independent research. Because the problems in the chapter are intended to be illustrative of the basic concepts introduced, many of the simpler ones may not do full justice to the specific situation being described. One particular conceptual shortcoming that characterizes most of the problems is that they do not incorporate any behavioral theory of government—that is, they implicitly assume that governments will undertake the efficient solution (i.e., a Pigovian tax) when it is called for. In discussion, students might be asked whether that is a reasonable assumption and how the theory might be modified to take actual government incentives into account. Some of the material in Chapter 20 might serve as additional background to such a discussion.

Comments on Problems

20.1 An example of a Pigovian tax on output. Instructors may wish to supplement this with a discussion of alternative ways to bring about the socially optimal reduction in output.

20.2 A simple example of the externalities involved in the use of a common resource. The allocational problem arises because average (rather than marginal) productivities are equated on the two lakes. Although an optimal taxation approach is examined in the problem, students might be asked to investigate whether private ownership of Lake *X* would achieve the same result.

20.3 Another example of externalities inherent in a common resource. This question poses a nice introduction to discussing “compulsory unitization” rules for oil fields and, more generally, for discussing issues in the market’s allocation of energy resources.

20.4 This is a descriptive problem involving externalities, now in relation to product liability law. For a fairly complete analysis of many of the legal issues posed here, see S. Shavell, *Economic Analysis of Accident Law*.

20.5 This is another discussion question that asks students to think about the relationship between various types of externalities and the choice of contract type. The Cheung article on sharecropping listed in the Suggested Readings for Chapter 20 provides a useful analysis of some of the issues involved in this question.

20.6 An illustration of the second-best principle to the externality issue. Shows that the ability of a Pigovian tax to improve matters depends on the specific way in which the market is organized.

20.7 An algebraic public goods problem in which students are asked to sum demand curves vertically rather than horizontally.

20.8 An algebraic example of the efficiency conditions that must hold when there is more than one public good in an economy.

20.9 Another public goods problem. In this case, the formulation is more general than in Problem 20.7 because there are assumed to be two goods and many (identical) individuals. The problem is fairly easy if students begin by developing an expression for the *RPT* and for the *MRS* for each individual and then apply Equation 20.40.

20.10 This problem asks students to generalize the discussions of Nash and Lindahl equilibria in public goods demand to n individuals. In general, inefficiencies are greater with *n* individuals than with only two.

Solutions

20.1 a. $MC = .4q$ $p = \$20$

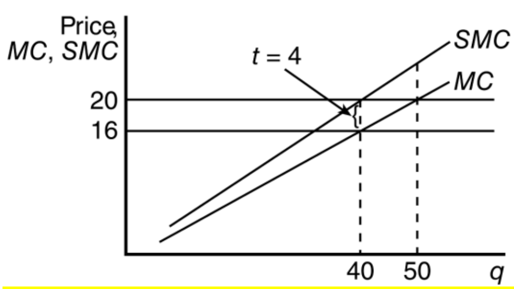
Set $p = MC$ $20 = .4q$ $q = 50.$

b. $SMC = .5q$

Set $p = SMC$ $20 = .5q$ $q = 40.$

At the optimal production level of $q = 40$, the marginal cost of production is $MC = .4q = .4(40) = 16$, so the excise tax $t = 20 - 16 = \$4$.

c.



20.2 a. $F^x = 10l_x - 0.5\, l_x^2 \quad F^y = 5l_y$

First, show how total catch depends on the allocation of labor.

$L_x + l_y = 20 \qquad l_y = 20 - l_x$

$F^T = F^x + F^y$

$F^T = 10l_x - .5\,l_x^2 + 5\,(20 - l_x)$

$\quad = 5l_x - 0.5l_x^2 + 100.$

Equating the average catch on each lake gives

$$\frac{F^x}{l_x} = \frac{F^y}{l_y} : 10 - 0.5\,l_x = 5 \qquad l_x = 10, l_y = 10$$

$F^T = 50 – 0.5(100) + 100 = 100.$

b. max $F^T: \ 5l_x - 0.5\,l_x^2 + 100$

$$\frac{dF^T}{dl_x} = 5 - l_x = 0 \qquad l_x = 5, l_y = 15, F^T = 112.5$$

c. $F^x_{\text{case 1}} = 50$ average catch = 50/10 = 5

$F^x_{\text{case 2}} = 37.5$ average catch = 37.5/5 = 7.5

License fee on Lake X should be = 2.5

d. The arrival of a new fisher on Lake X imposes an externality on the fishers already there in terms of a reduced average catch. Lake X is treated as common property here. If the lake were private property, its owner would choose L_X to maximize the total catch less the opportunity cost of each fisher (the 5 fish he/she can catch on Lake Y). So the problem is to maximize $F^X - 5l_x$ which yields $l_x = 5$ as in the optimal allocation case.

20.3 *AC* = *MC* = 1000/well

a. Produce where revenue/well = 1000 = 10q = 5000 –10n. n = 400. There is an externality here because drilling another well reduces output in *all* wells.

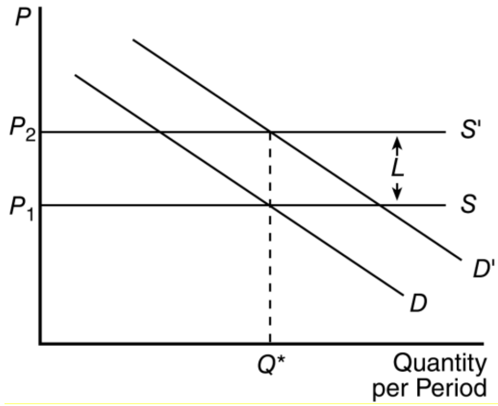
b. Produce where *MVP* = *MC* of well. Total value:

$5000n - 10n^2$. $MVP = 5000 - 20n = 1000$. $n = 200$.

Let tax = x. Want revenue/well –x = 1000 when n = 200. At n = 200, average revenue/well = 3000.

So charge x = 2000.

20.4 Under *caveat emptor*, buyers would assume all losses. The demand curve under such a situation might be given by *D*. Firms (which assume no liability) might have a horizontal long-run supply curve of *S*. A change in liability assignment would shift both supply and demand curves. Under *caveat vendor*, losses (of amount *L*) would now be incurred by firms, thereby shifting the long-run supply curve to *S*'.



Individuals now no longer have to pay these losses and their demand curve will shift upward by *L* to *D*'. In this example, then, market price rises from P_1 to P_2 (although the real cost of owning the good has not changed), and the level of production stays constant at Q^*. Only if there were major information costs associated with either the *caveat emptor* or *caveat vendor* positions might the two give different allocations. It is also possible that *L* may be a function of liability assignment (the moral hazard problem), and this would also cause the equilibria to differ.

20.5 There is considerable literature on this question, and a good answer should only be expected to indicate some of the more important issues. Aspects of what might be mentioned include

A. specific services provided by landlords and tenants under the contracts.

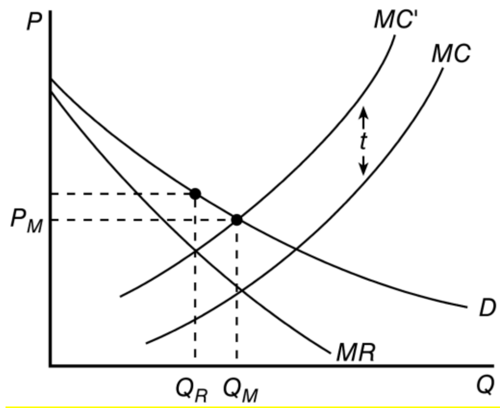
B. the risks inherent in various types of contracts, who bears these risks, and how is that likely to affect demand or supply decisions.

C. costs of gathering information before the contract is concluded, and of enforcing the contract's provisions.

D. the incentives provided for tenant and landlord behavior under the contracts (for example, the incentives to make investments in new production techniques or to alter labor supply decisions).

E. "noneconomic" aspects of the contracts such as components of landlords' utility functions or historical property relationships.

20.6



In the diagram the untaxed monopoly produces Q_M at a price of P_M. If the marginal social cost is given by *MC'*, Q_M is, in fact, the optimal production level. A per-unit tax of *t* would cause the monopoly to produce output Q_R, which is below the optimal level. Since a tax will always cause such an output restriction, the tax may improve matters only if the optimal output is less than Q_M, and even then, in many cases it will not.

20.7 a. To find the total demand for mosquito control, demand curves must be summed vertically. Letting *Q* be the total quantity of mosquito control (which is equally consumed by the two individuals), the individuals' marginal valuations are

$P = 100 – Q$ (for a)

$P = 200 – Q$ (for b).

Hence, the total willingness to pay is given by $300 – 2Q$.

Setting this equal to *MC* (= 120) yields optimal $Q = 90$.

b. In the private market, price will equal $MC = 120$. At this price (a) will demand 0, (b) will demand 80. Hence, output will be less than optimal.

c. A tax price of 10 for (a) and 110 for (b) will result in each individual demanding $Q = 90$ and tax collections will exactly cover the per-unit cost of mosquito control.

20.8 a. For each public good (y_i, $i = 1, 2$) the *RPT* of the good for the private good (a) should equal the sum of individuals' *MRS*'s for the goods:

$$RPT = (y_i \text{ for } x) = \sum_{1}^{n} MRS \ (y_i \text{ for } x).$$

b. For the two public goods (y_1 and y_2), the *RPT* between the goods should equal the ratio of the sums of the marginal utilities for each public good:

$$RPT\ (y_1 \text{ for } y_2) = \frac{\sum_n \mathrm{M}U_i\ (y_1)}{\sum_n MU_i\ (y_2)}.$$

20.9 a. The solution here requires some assumption about how individuals form their expectations about what will be purchased by others. If each assumes he or she can be a free rider, *y* will be zero as will be each person's utility.

b. Taking total differential of production possibility frontier.

$2x\,dx + 200y\,dy = 0$ gives

$$RPT = -\frac{dx}{dy} = \frac{200y}{2x} = \frac{100y}{x}$$

Individual $MRS_i = \frac{MU_y}{MU_x} = \frac{0.5\sqrt{\frac{x_i}{y}}}{0.5\sqrt{\frac{y}{x_i}}} = \frac{x_i}{y} = \frac{x/100}{y}$

For efficiency require sum of *MRS* should equal *RPT*

$\sum_i MRS_i = \frac{x}{y}$. Hence, $\frac{x}{y} = \frac{100y}{x}$ $x = 10y$.

Using production possibility frontier yields

$200y^2 = 5000$

$y = 5$

$x = 50$ $x/100 = 0.5$

Utility = $\sqrt{2.5}$.

Ratio of per-unit tax share of *y* to the market price of *x* should be equal to the $MRS = \frac{x_i}{y} = \frac{1}{10}$.

20.10 a. The condition for efficiency is that $\sum_1^n MRS_i = RPT$. The fact that the MRS's are summed captures the assumption that each person consumes the same amount of the nonexclusive public good. The fact that the RPT is independent of the level of consumers shows that the production of the good is nonrival.

b. As in Equation 20.41, under a Nash equilibrium each person would opt for a share under which $MRS_i = RPT$ implies a much lower level of public good production than is efficient.

c. Lindahl Equilibrium requires that $\alpha_i = MRS_i / RPT$ and $\sum \alpha_i = 1.0$. This would seem to pose even greater informational difficulties than in the two-person case.

CHAPTER 21

POLITICAL ECONOMICS

The problems in this final chapter are of two general types. First are four problems in traditional welfare economics (Problems 21.1–21.3 and 21.5) that illustrate various issues that arise in comparing utility among individuals. These are rather similar to the problems in Chapter 12. The other six problems in the chapter concern public choice theory.

Comments on Problems

21.1 A problem utilizing two very simple utility functions to show how none of several differing welfare criteria seems necessarily superior to all the others. This clearly illustrates the basic dilemma of traditional welfare economics.

21.2 This problem examines the Scitovsky bribe criterion for judging welfare improvements. Although the criterion as a general principle is not widely accepted, the notion of “bribes” in public policy discussions is still quite prevalent (for example, in connection with trade adjustment policies).

21.3 Shows how to integrate production into the utility possibility frontier construction. In the example given here, the frontiers are concentric ellipses so the Pareto criterion suggests choosing the one that is furthest from the origin. The choice is, however, ambiguous if the frontiers intersect.

21.4 Illustrates the “irrelevant alternative” assumption in the Arrow theorem.

21.5 A further examination of welfare criteria that focuses on Rawls’ uncertainty issues. Shows that the results derived from a Rawls’ “initial position” depend crucially on the strategies individuals adopt in risky situations.

21.6 Further examination of the Arrow theorem and of how contradictions can arise in fairly simple situations.

21.7 A simple problem focusing on an individual’s choice for the parameters of an unemployment insurance policy. The problem would need to be generalized to provide testable implications about voting (see the *Persson and Tabellini* reference).

21.8 A problem in rent seeking. The main point is to differentiate between the allocational harm of monopoly itself and the transfer nature of rent-seeking expenditures.

21.9 A discussion question concerning voter participation.

21.10 An alternative specification for probabilistic voting that also yields desirable normative consequences.

Solutions

21.1 200 pounds $U_1 = \sqrt{f_1}$ $U_2 = \frac{1}{2}\sqrt{f_2}$

a. 100 pounds each $U_1 = 10$, $U_2 = 5$

b. $\sqrt{f_1} = \frac{1}{2}\sqrt{f_2}$ $f_1 = \frac{1}{4} f_2$

$f_1 = 40$ $f_2 = 160$

c. $U_1 + U_2 = \sqrt{f_1} + \frac{1}{2}\sqrt{200 - f_1}$

$$\frac{1}{2} f_1^{-0.5} = \frac{1}{4}(200 - f_1)^{-0.5}$$

$f_1 = 160$, $f_2 = 40$

d. $U_2 \geq 5$, best choice is $U_2 = 5$

$f_2 = 100$, $f_1 = 100$.

e. $W = U_1^{0.5} U_2^{0.5} = f_1^{技} \cdot \frac{1}{\sqrt{2}} f_2 = \frac{1}{\sqrt{2}} f_1^{技}(200 - f_1)$

$$\frac{\partial W}{\partial f_1} = \frac{1}{\sqrt{2}} f_1^{?} [-\frac{1}{4}(200 - f_1)^{- /4}] + \frac{1}{\sqrt{2}}\frac{1}{4} f_1^{-3/4}(200 - f_1) = 0$$

$$f_1^{?}(200 - f_1)^{- /4} = f_1^{-3/4}(200 - f_1)$$

$f_1 = 200 - f_1$ $f_1 = 100$, $f_2 = 100$.

21.2 If compensation is not actually made, the bribe criterion amounts to assuming that total dollars and total utility are commensurable across individuals. As an example, consider:

	Income in State A	**Income in State B**
Individual 1	100,000	110,000
Individual 2	5,000	0

State B is "superior" to State A in that Individual 1 *could* bribe Individual 2. But, in the absence of compensation actually being made, it is hard to argue that State B is better.

21.3 Pareto efficiency requires $MRS_1 = MRS_2$

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{\bar{y} - y_1}{\bar{x} - x_1} .$$

Hence, all efficient allocations have

$$y_1 = \alpha\bar{y} \quad x_1 = \alpha\bar{x}$$

$$y_2 = (1-\alpha)\bar{y} \quad x_2 = (1-\alpha)\bar{x} .$$

$$U_1^2 = \alpha^2\overline{xy} \quad U_2^2 = (1-\alpha)^2\overline{xy}$$

$$(U_1 + U_2)^2 = U_1^2 + 2U_1U_2 + U_2^2 = \alpha^2\overline{xy} + 2(\alpha)(1-\alpha)\overline{xy} + (1-\alpha)^2\overline{xy} = \overline{xy}$$

a. If $\bar{y} = 10, \bar{x} = 160$

   Utility Frontier is $(U_1 + U_2)^2 = 1600$.

b. $\bar{y} = 30 \quad \bar{x} = 120$

   $(U_1 + U_2)^2 = 3600$

c. Maximize $\overline{XY}$ subject to $\bar{X} + 2\bar{Y} = 180$ yields $\bar{x} = 90 \quad \bar{y} = 45$

   $(U_1 + U_2)^2 = 4050$.

d. In this problem, the utility possibility frontiers do not intersect, so there is no ambiguity in using the Pareto criterion. If they did intersect, however, one would want to use an outer envelope of the frontiers.

21.4 7 individuals with states A, B, C. Votes are

A	B	C
3	2	2

If C is not available, let both C votes go to B.

A	B
3	4

This example is quite reasonable: it implies that Arrow's axiom is rather restrictive.

21.5 a. *D*

b. *E*, $E(U) = .5(30) + .5(84) = 57$

c. $E(U) = .6(L) + .4(H)$

   $EU_A = 50$, $EU_B = 52$, $EU_C = 48.6$, $EU_D = 51.5$, $EU_E = 50$.

   So choose B.

d. max $E(U) - |U_1 - U_2|$

values: A: 50 – 0 = 50 C: 49.5 – 9 = 40.5 E: 57 – 54 = 3

B: 55 – 30 = 26 D: 51.75 – 2.5 = 49.26

So choose A.

e. It shows that a variety of different choices might be made depending on the criteria being used.

21.6 Suppose preferences are as follows:

	Individual		
	1	2	3
Preference	C	A	B
	A	B	C
	B	C	A

a. Under majority rule, *APB* (where *P* means "is socially preferred to"), *BPC,* but *CPA*. Hence, the transivity axiom is violated.

b. Suppose Individual 3 is very averse to *A* and reaches an agreement with Individual 1 to vote for *C* over *B* if Individual 1 will vote for *B* over *A*. Now, majority rule results in *CPA, CPB,* and *BPA*. The final preference violates the nondictatorship assumption since *B* is preferred to *A* only by Individual 3.

c. With point voting, each option would get six votes, so *AIBIC*. But that result can be easily overturned by introducing an "irrelevant alternative" (*D*).

21.7 a. So long as this utility function exhibits diminishing marginal utility of income, this person will opt for parameters that yield $y_1 = y_2$. Here that requires $w(1 - t) = b$. Inserting this into the governmental budget constraint produces $uw(1 - t) = tw(1 - u)$ which requires $u = t$.

b. A change in *u* will change the tax rate by an identical amount.

c. The solutions in parts a and b are independent of the risk aversion parameter, δ.

21.8 a. Since $p = -q/100 + 2$, $MR = -q/50 + 2$

$MR = MC$ when $q = 75$, $p = 1.25$, $\pi = 56.25$.

The firm would be willing to pay up to this amount to obtain the concession (assuming that competitive results would otherwise obtain).

b. The bribes are a transfer, not a welfare cost.

c. The welfare loss is the deadweight loss from monopolization of this market, which here amounts to 28.125.

21.9 An essay on this topic would stress that free riding may be a major problem in elections where voters perceive that the marginal gain from voting may be quite small. If such voters are systematically different from other voters, candidates will recognize this fact and tailor their platforms to those who vote rather than to the entire electorate. The effect would be ameliorated by the extent to which platforms can affect voter participation itself.

21.10 Candidate 1's problem is to chose θ_1, to maximize

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} f_i(U_i(\theta_{1i})/U_i(\theta_{2i}) \text{ subject to } \sum_{i=1}^{n} \theta_{1i} = 0.$$

The first order conditions for a maximum are $f_i^{'} U_i^{'}/U_i(\theta_{2i}^{*}) = \lambda$ for all $i = 1 \ldots n$.

Assuming $f_i^{'}$ is the same for all individuals, this yields $U_i^{'}/U_i(\theta_{2i}^{*}) = k\lambda$ for $i = 1 \ldots n$.

In words, the candidate should equate the ratio of the marginal utilities of any two voters $(U_i^{'}/U_j^{'})$ to the ratio of their total utilities (U_i/U_j). Since each candidate follows this strategy, they will adopt the strategies that would maximize the Nash Function, *SWF*.